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Abstract

In this paper we address the issue of the Gribov copies in $SU(N)$, $N > 2$, Euclidean Yang–Mills theories quantized in the maximal Abelian gauge. A few properties of the Gribov region in this gauge are established. Similar to the case of $SU(2)$, the Gribov region turns out to be convex, bounded along the off-diagonal directions in field space, and unbounded along the diagonal ones. The implementation of the restriction to the Gribov region in the functional integral is discussed through the introduction of the horizon function, whose construction will be outlined in detail. The influence of this restriction on the behavior of the gluon and ghost propagators of the theory is also investigated together with a set of dimension 2 condensates.

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1. Introduction

In his seminal work [1], Gribov pointed out that the quantization of Yang–Mills theories through the Faddeev–Popov method is plagued by the existence of the Gribov copies. Even if one imposes a subsidiary gauge-fixing condition in an attempt to remove the gauge redundancy, there still exist field configurations belonging to the same gauge orbit which fulfill such a condition, i.e. there will be equivalent field configurations, or copies, in the gauge-fixed theory. As a consequence, the functional measure in the Feynman path integral becomes ill-defined.

In order to circumvent this problem, Gribov suggested [1] that the domain of integration in the Feynman path integral should be restricted to a certain region, known as the Gribov

region, which was supposed to contain inequivalent field configurations only. Nowadays, it is known that the Gribov region is not completely free from Gribov copies, i.e. additional copies still exist within the Gribov region [2, 3]. In order to avoid the presence of these additional copies, a further restriction to a smaller region, known as the fundamental modular region, should be implemented [2, 3], though a procedure of effectively achieving the restriction to the fundamental modular region is still beyond our present capabilities. Therefore, in the following, we shall limit ourselves to investigating the Gribov region.

In [1], the Landau gauge was employed in order to illustrate several aspects related to the existence of the Gribov copies. In subsequent works [3–6], properties of the Gribov region in this gauge were established. In particular, in [7, 8], Zwanziger was able to show that the restriction to the Gribov region could be achieved by adding to the Faddeev–Popov action an additional nonperturbative term called the horizon function. Remarkably, the resulting action, known as the Gribov–Zwanziger action, can be cast in the local form and proven to be multiplicatively renormalizable [7–11]. More recently, in [12–15], dynamical effects related to the condensation of local dimension 2 operators have been accommodated in the Gribov–Zwanziger action in a way that is compatible with locality and renormalizability, giving rise to the so-called refined Gribov–Zwanziger model. As far as the behavior of the gluon and ghost propagators is concerned, the refined model yields a positivity-violating gluon propagator which is suppressed in the infrared region and which does not vanish at zero momentum, while giving a ghost propagator which is no longer enhanced in the infrared region, behaving essentially as $\frac{1}{k^2}$ for $k \approx 0$. Such behavior, referred to as the decoupling or massive solution, has also been obtained within the context of the Schwinger–Dyson equations [16–19]. So far, the behavior of the gluon and ghost propagators in the infrared region obtained from the refined model seems to be in agreement with recent lattice data in the Landau gauge [20–25], though we have to underline that no unanimous consensus on this matter has yet been reached, see e.g. [19, 26, 27] for other possible positions. At present, the issue of the infrared behavior of the gluon and ghost propagators in the Landau gauge is the object of a rather interesting and intensive debate.

The effects stemming from the existence of the Gribov copies turn out to be relevant in the nonperturbative infrared region and might play an important role for color confinement in QCD. Moreover, this is a central issue toward a correct quantization of Yang–Mills theories. As pointed out in [28], the existence of the Gribov copies is in fact a general feature of the gauge-fixing procedure. Besides the Landau gauge, the issue of the Gribov copies has been addressed in other gauges such as the Coulomb gauge [29] and the linear covariant gauge [30], which has the Landau gauge as a particular case. Concerning the maximal Abelian gauge, only the particular case of $SU(2)$ has been discussed so far [31, 32].

In the present work we shall consider the case of Euclidean $SU(N)$, $N > 2$, Yang–Mills theories in the maximal Abelian gauge. Let us also recall that this gauge is suitable for the study of the so-called Abelian dominance hypothesis [33–35], which is one of the main ingredient of the dual superconductivity mechanism for color confinement [36–38], according to which Yang–Mills theories in the low energy region should be described by an effective Abelian theory in the presence of monopoles. A dual Meissner effect arising as a consequence of the condensation of these magnetic charges might give rise to the formation of flux tubes which confine the quarks. The maximal Abelian gauge also displays the important property of possessing a lattice formulation [39–41], while being a renormalizable gauge in the continuum [42–45].

As already mentioned, several works have been devoted to investigate the effects of the Gribov copies in the maximal Abelian gauge in the case of $SU(2)$. A study of the influence of the Gribov copies on the two-point gluon and ghost correlation functions was reported in [32].

These propagators have also been investigated within the Schwinger–Dyson framework, see [46]. In [47], general properties of the Gribov region in this gauge were established. In [48], the horizon function was obtained under the requirements of localizability and renormalizability. Finally, the inclusion of a set of local dimension 2 operators in the presence of the horizon function was presented in [49], giving rise to a model which is analogous to the refined Gribov–Zwanziger of the Landau gauge. It is worth pointing out that the resulting behavior of the tree-level gluon and ghost propagators obtained in [49] are in good agreement with the most recent lattice numerical simulations [41] done in the case of $SU(2)$.

In the present work, we attempt to generalize the entire program outlined in the series of papers [32, 47–49] to $SU(N)$ Yang–Mills theories in the maximal Abelian gauge. The general case of $SU(N)$, $N > 2$, is interesting for, at least, two reasons. Firstly, as the symmetry group of QCD is $SU(3)$, the case of $N = 3$ is of particular interest. Secondly, as we shall see, for $N > 2$, the Faddeev–Popov operator of the maximal Abelian gauge possesses an additional term which is absent in the case of $SU(2)$. This term will induce modifications on the behavior of the correlation functions. For $N = 2$, only the diagonal components of the gluon propagator are affected by the restriction to the Gribov region. However, for $N > 2$, the off-diagonal components are also affected through a term proportional to $(N - 2)$. This feature might raise interest from the lattice community. A numerical study of the gluon propagator in the case of $SU(3)$ is, to our knowledge, still lacking.

The paper is organized as follows. In section 1, the Faddeev–Popov quantization in the maximal Abelian gauge is shortly reviewed. In section 2, we introduce the Gribov region Ω and we establish some of its properties. This region turns out to be convex, unbounded in all diagonal directions in field space and bounded along the off-diagonal directions. Section 3 is devoted to the proof of a useful statement which ensures that for any field configuration belonging to the Gribov region Ω and located near its boundary, there is a Gribov copy, close to the boundary of Ω, however located outside of Ω. This statement, originally proven by Gribov in the case of the Landau gauge [1], provides us a support for restricting the domain of integration in the functional integral to the region Ω. The effective implementation of such a restriction will be the main subject of section 4. In section 5, in order to go beyond Gribov's quadratic approximation, we introduce the horizon function. Although in the case of $SU(N)$ several possible candidates for the horizon function can be written down, only one term turns out to be selected by the requirements of being localizable, of reproducing the horizon function already introduced in the case of $SU(2)$ and of coinciding with the expression obtained from Gribov's no pole condition [1]. In section 6, a local action containing a suitable set of dimension 2 operators chosen in such a way as to preserve the symmetry content of the theory is presented. The resulting gluon and ghost propagators will be worked out. Finally, in section 7, we collect our conclusion and discuss some perspectives.

2. The maximal Abelian gauge

2.1. Some useful definitions

In order to introduce the maximal Abelian gauge-fixing condition, one makes use of the following decomposition for the gauge field $A_\mu(x)$:

$$A_\mu = A_\mu^A T^A = A_\mu^a T^a + A_\mu^i T^i, \tag{1}$$

where T^A are the $(N^2 - 1)$ generators of $SU(N)$, while T^a and T^i stand, respectively, for the off-diagonal and diagonal generators. There are $N(N - 1)$ off-diagonal generators and $(N - 1)$ diagonal ones. The diagonal generators give rise to the Abelian subgroup $U(1)^{N-1}$,

which is known as the Cartan subgroup. Here, we have adopted three sets of indices, namely

$$
\begin{aligned}
&A, B, C, D, E, \ldots \in \{1, \ldots, (N^2 - 1)\}, \\
&a, b, c, d, e, \ldots \in \{1, \ldots, N(N - 1)\}, \\
&i, j, k, l, \ldots \in \{1, \ldots, (N - 1)\},
\end{aligned}
\tag{2}
$$

where the capital Latin indices $\{A, B, C, D, \ldots\}$ are the usual $SU(N)$ group indices. The off-diagonal indices are represented by $\{a, b, c, d, e, \ldots\}$ and run from 1 to $N(N - 1)$, while the diagonal indices are given by $\{i, j, k, l, \ldots\}$ running from 1 to $(N - 1)$. The commutation relations between the generators T^A, T^a, T^i are given by

$$
\begin{aligned}
&[T^A, T^B] = \mathrm{i} f^{ABC} T^C, \\
&[T^a, T^b] = \mathrm{i} f^{abc} T^c + \mathrm{i} f^{abi} T^i, \\
&[T^a, T^i] = -\mathrm{i} f^{abi} T^b, \\
&[T^i, T^j] = 0,
\end{aligned}
\tag{3}
$$

where f^{ABC} denote the structure constants of $SU(N)$. From the last equation of (3) we conclude that $f^{aij} = f^{ijk} = 0$. The remaining nonvanishing structure constants f^{abc} and f^{abi} are totally antisymmetric by the exchange of indices and obey the following identities[3]:

$$
\begin{aligned}
0 &= f^{abi} f^{bcj} + f^{abj} f^{bic}, \\
0 &= f^{abc} f^{cdi} + f^{adc} f^{cib} + f^{aic} f^{cbd}, \\
0 &= f^{abc} f^{cde} + f^{abi} f^{ide} + f^{adc} f^{ceb} + f^{adi} f^{ieb} + f^{aec} f^{cbd} + f^{aei} f^{ibd}.
\end{aligned}
\tag{4}
$$

The off-diagonal and diagonal components of the gauge field transform under an infinitesimal gauge transformation as

$$
\delta A_\mu^a = -\big(D_\mu^{ab} \omega^b + g f^{abc} A_\mu^b \omega^c + g f^{abi} A_\mu^b \omega^i\big),
\tag{5}
$$

$$
\delta A_\mu^i = -\big(\partial_\mu \omega^i + g f^{abi} A_\mu^a \omega^b\big).
\tag{6}
$$

Here, D_μ^{ab} is the covariant derivative with respect to the diagonal components

$$
D_\mu^{ab} = \delta^{ab} \partial_\mu - g f^{abi} A_\mu^i,
\tag{7}
$$

and (ω^a, ω^i) are the gauge parameters. The Yang–Mills action can also be written in terms of these diagonal and off-diagonal components as

$$
S_{\mathrm{YM}} = \frac{1}{4} \int \mathrm{d}^4 x \big(F_{\mu\nu}^a F_{\mu\nu}^a + F_{\mu\nu}^i F_{\mu\nu}^i\big),
\tag{8}
$$

with

$$
\begin{aligned}
F_{\mu\nu}^a &= D_\mu^{ab} A_\nu^b - D_\nu^{ab} A_\mu^b + g f^{abc} A_\mu^b A_\nu^c, \\
F_{\mu\nu}^i &= \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g f^{abi} A_\mu^a A_\nu^b.
\end{aligned}
\tag{9}
$$

---

[3] These identities follow from the Jacobi identity

$$
f^{ABC} f^{CDE} + f^{ADC} f^{CEB} + f^{AEC} f^{CBD} = 0.
$$

2.2. Describing the maximal Abelian gauge condition

Since we have split the gauge field $A_\mu(x)$ into two components, a diagonal (or Abelian) one represented by $A_\mu^i(x)$ and an off-diagonal one given by $A_\mu^a(x)$, we can now choose to fix the gauge invariance of expression (8) by imposing a gauge condition on each component of the gauge field. For the off-diagonal components $A_\mu^a(x)$, one imposes the condition

$$D_\mu^{ab} A_\mu^b = \partial_\mu A_\mu^a - g f^{abi} A_\mu^i A_\mu^b = 0, \tag{10}$$

while for the diagonal ones one requires

$$\partial_\mu A_\mu^i = 0. \tag{11}$$

Note the nonlinearity in the gauge fixing of the off-diagonal components, equation (10). This particular choice of gauge fixing is supported by the interesting property that it follows from the requirement that the auxiliary functional

$$\mathcal{F}[A] = \int d^4x \, A_\mu^a A_\mu^a \tag{12}$$

is stationary with respect to the local gauge transformations (6). The nonlinear condition (10) still allows for a remaining local $U(1)^{N-1}$ invariance which is removed by imposing a gauge condition on the diagonal components, for which a Landau-type condition is usually chosen, as expressed by equation (11). Conditions (10) and (11) are referred to as the maximal Abelian gauge.

For further use, let us also display here conditions (10) and (11) in momentum space. By performing a Fourier transformation, one straightforwardly obtains

$$k_\mu A_\mu^a(k) = \mathrm{i} g f^{abi} \int \frac{d^4 p}{(2\pi)^4} A_\mu^i(k-p) A_\mu^b(p),$$
$$k_\mu A_\mu^i(k) = 0. \tag{13}$$

2.3. The Faddeev–Popov quantization

The gauge-fixing term, which naturally arises from the Faddeev–Popov quantization method, assumes the following form in the case of the maximal Abelian gauge:

$$
\begin{aligned}
S_{\text{MAG}} &= s \int d^4x \left(\bar{c}^a D_\mu^{ab} A_\mu^b + \bar{c}^i \partial_\mu A_\mu^i \right) \\
&= \int d^4x \Big[\mathrm{i} b^a D_\mu^{ab} A_\mu^b - \bar{c}^a \mathcal{M}^{ab} c^b - g f^{abc} \big(D_\mu^{ad} A_\mu^d \big) \bar{c}^b c^c - g f^{abi} \big(D_\mu^{ac} A_\mu^c \big) \bar{c}^b c^i \\
&\quad + \mathrm{i} b^i \partial_\mu A_\mu^i + \bar{c}^i \partial_\mu \big(\partial_\mu c^i + g f^{abi} A_\mu^a c^b \big) \Big].
\end{aligned}
\tag{14}
$$

In this expression, the auxiliary fields (b^a, b^i) are the Lagrange multipliers enforcing the maximal Abelian gauge-fixing conditions (10), (11); (c^a, c^i) are the Faddeev–Popov ghost fields, while $(\bar{c}^a, \bar{c}^i)$ are the anti-ghost fields; $\mathcal{M}^{ab}$ stands for the Faddeev–Popov operator, given by

$$\mathcal{M}^{ab} = -D_\mu^{ac} D_\mu^{cb} - g f^{acd} A_\mu^c D_\mu^{db} - g^2 f^{aci} f^{bdi} A_\mu^c A_\mu^d; \tag{15}$$

and finally, s is the nilpotent BRST operator acting on the fields as

$$
\begin{aligned}
sA_\mu^a &= -\big(D_\mu^{ab} c^b + g f^{abc} A_\mu^b c^c + g f^{abi} A_\mu^b c^i \big), & sA_\mu^i &= -\big(\partial_\mu c^i + g f^{abi} A_\mu^a c^b \big), \\
sc^a &= g f^{abi} c^b c^i + \frac{g}{2} f^{abc} c^b c^c, & sc^i &= \frac{g}{2} f^{abi} c^a c^b, \\
s\bar{c}^a &= \mathrm{i} b^a, & s\bar{c}^i &= \mathrm{i} b^i, \\
sb^a &= 0, & sb^i &= 0.
\end{aligned}
\tag{16}
$$

The nonlinearity of the gauge fixing implies that an additional self-interacting ghost term is required in order to ensure renormalizability [42–45]. This additional term is defined as follows:

$$
\begin{aligned}
S_\alpha &= -\frac{\alpha}{2} s \int \mathrm{d}^4x \left(\mathrm{i}\bar{c}^a b^a - g f^{abi} \bar{c}^a \bar{c}^b c^i - \frac{g}{2} f^{abc} c^a \bar{c}^b \bar{c}^c \right) \\
&= \frac{\alpha}{2} \int \mathrm{d}^4x \left(b^a b^a + 2\mathrm{i}g f^{abi} b^a \bar{c}^b c^i + i g f^{abc} b^a \bar{c}^b c^c + \frac{g^2}{2} f^{abi} f^{cdi} \bar{c}^a \bar{c}^b c^c c^d \right. \\
&\left. \quad + \frac{g^2}{2} f^{abc} f^{adi} \bar{c}^b \bar{c}^c c^d c^i + \frac{g^2}{4} f^{abc} f^{ade} \bar{c}^b \bar{c}^c c^d c^e \right).
\end{aligned}
\tag{17}
$$

Note that this term is proportional to a gauge parameter α and thus the original gauge fixing is recovered when α is set to zero. However, the limit $\alpha \to 0$ has to be taken after the removal of the ultraviolet divergences. In fact, some of the terms proportional to α would reappear due to radiative corrections, even if $\alpha = 0$ [42–45]. Moreover, the action

$$
S = S_{\mathrm{YM}} + S_{\mathrm{MAG}} + S_\alpha
\tag{18}
$$

is multiplicatively renormalizable to all orders of the perturbation theory [42–45].

To conclude this section, let us make a few remarks on the partition function of the theory, namely

$$
\mathcal{Z} = \int \mathcal{D}A^a \mathcal{D}A^i \mathcal{D}b^a \mathcal{D}b^i \mathcal{D}c^a \mathcal{D}\bar{c}^a \mathcal{D}c^i \mathcal{D}\bar{c}^i \, \mathrm{e}^{-S[A,b,c,\bar{c}]},
\tag{19}
$$

where S is given by (18). Now, taking the limit $\alpha \to 0$ and integrating over the Lagrange multipliers (b^a, b^i), one obtains

$$
\mathcal{Z} = \int \mathcal{D}A^a \mathcal{D}A^i \mathcal{D}c^a \mathcal{D}\bar{c}^a \mathcal{D}c^i \mathcal{D}\bar{c}^i \, \delta\big(D_\mu^{ab} A_\mu^b\big) \delta\big(\partial_\mu A_\mu^i\big) \, \mathrm{e}^{-S_{\mathrm{YM}} + \int \mathrm{d}^4x [\bar{c}^a \mathcal{M}^{ab} c^b - \bar{c}^i \, \partial_\mu(\partial_\mu c^i + g f^{abi} A_\mu^a c^b)]}.
\tag{20}
$$

To deal with the diagonal ghosts $(c^i, \bar{c}^i)$, we perform the following change of variables:

$$
c^i \to \xi^i = c^i + g f^{abi} \frac{\partial_\mu}{\partial^2} A_\mu^a c^b, \qquad \bar{c}^i \to \bar{\xi}^i = \bar{c}^i,
\tag{21}
$$

with all other fields unchanged. Being linear in $(c^i, \bar{c}^i)$, this change of variables leads to a Jacobian which is field independent. Thus, we can verify that

$$
\bar{c}^i \, \partial_\mu\big(\partial_\mu c^i + g f^{abi} A_\mu^a c^b\big) \to \bar{\xi}^i \, \partial^2 \xi^i.
\tag{22}
$$

Therefore, for the partition function we have

$$
\begin{aligned}
\mathcal{Z} &= \left(\int \mathcal{D}\xi^i \mathcal{D}\bar{\xi}^i \, \mathrm{e}^{-\int \mathrm{d}^4x \, \bar{\xi}^i \partial^2 \xi^i} \right) \int \mathcal{D}A^a \mathcal{D}A^i \mathcal{D}c^a \mathcal{D}\bar{c}^a \, \delta\big(D_\mu^{ab} A_\mu^b\big) \delta\big(\partial_\mu A_\mu^i\big) \, \mathrm{e}^{-S_{\mathrm{YM}} + \int \mathrm{d}^4x \, \bar{c}^a \mathcal{M}^{ab} c^b} \\
&= \mathcal{N} \int \mathcal{D}A^a \mathcal{D}A^i \mathcal{D}c^a \mathcal{D}\bar{c}^a \, \delta\big(D_\mu^{ab} A_\mu^b\big) \delta\big(\partial_\mu A_\mu^i\big) \, \mathrm{e}^{-S_{\mathrm{YM}} + \int \mathrm{d}^4x \, \bar{c}^a \mathcal{M}^{ab} c^b},
\end{aligned}
\tag{23}
$$

with $\mathcal{N}$ being an irrelevant constant factor. Finally, integrating over the off-diagonal ghosts $(c^a, \bar{c}^a)$, it follows that

$$
\mathcal{Z} = \mathcal{N} \int \mathcal{D}A^a \mathcal{D}A^i \, \delta\big(D_\mu^{ab} A_\mu^b\big) \delta\big(\partial_\mu A_\mu^i\big) \det(\mathcal{M}^{ab}) \, \mathrm{e}^{-S_{\mathrm{YM}}}.
\tag{24}
$$

As we shall see later, expression (24) will be taken as the starting point for the implementation of the restriction of the domain of integration to the Gribov region Ω.

3. The Gribov region Ω and some of its properties

3.1. On the existence of Gribov copies in the maximal Abelian gauge

Before entering into the analysis of the Gribov region Ω, let us briefly discuss the existence of copies in the maximal Abelian gauge. In order to deal with this issue, let us first take a field configuration $\left(A_\mu^a, A_\mu^i\right)$ fulfilling the gauge conditions (10), (11). Then, we can ask if there exists a gauge transformed configuration, $\left(\widetilde{A}_\mu^a, \widetilde{A}_\mu^i\right)$, which fulfills the same gauge conditions as $\left(A_\mu^a, A_\mu^i\right)$, i.e.

$$D_\mu^{ab}(\widetilde{A})\widetilde{A}_\mu^b = 0 \qquad \partial_\mu \widetilde{A}_\mu^i = 0. \tag{25}$$

In the case of infinitesimal gauge transformations, this question can be answered in the affirmative if the following conditions can be verified for some value of the gauge parameters (ω^a, ω^i):

$$\mathcal{M}^{ab}(A)\omega^b = 0, \qquad -\partial_\mu\left(\partial_\mu \omega^i + g f^{abi} A_\mu^a \omega^b\right) = 0, \tag{26}$$

where $\mathcal{M}^{ab}$ is the Faddeev–Popov operator given by (15). Note that the diagonal parameter $\omega^i(x)$ is completely determined in terms of the off-diagonal quantities $\omega^a(x)$ and $A_\mu^a(x)$. In fact, from equations (26), it follows

$$\omega^i = -g f^{abi} \frac{1}{\partial^2} \partial_\mu \left(A_\mu^a \omega^b\right). \tag{27}$$

Thus, we can say that the condition for the existence of equivalent field configurations, or Gribov copies, relies on the existence of zero modes of the Faddeev–Popov operator $\mathcal{M}^{ab}(A)$. Explicit examples of such zero modes can be found in [31].

3.2. Determining the Gribov region Ω in the maximal Abelian gauge

The so-called Gribov region Ω in the maximal Abelian gauge can be defined in a way similar to the case of the Landau gauge [7, 8]. More precisely, the region Ω consists of all field configurations which are relative minima of the auxiliary functional (12), which is the same as to require that $\mathcal{M}^{ab}$ has to be positive, namely

$$\delta^2 \mathcal{F}[A] = 2 \int d^4x\, \omega^a \mathcal{M}^{ab}(A)\omega^b > 0 \quad \Rightarrow \quad \mathcal{M}^{ab}(A) > 0. \tag{28}$$

Thus, we can write

$$\Omega = \left\{ A_\mu^a, A_\mu^i \mid D_\mu^{ab} A_\mu^b = 0, \partial_\mu A_\mu^i = 0, \mathcal{M}^{ab}(A) > 0 \right\}. \tag{29}$$

The boundary ∂Ω of Ω is the region in field space where $\mathcal{M}^{ab}$ achieve its first vanishing eigenvalue. This boundary is often called the *first Gribov horizon* or simply the *Gribov horizon*. Note also that for very small values of the coupling constant g, corresponding to the perturbative regime, the Faddeev–Popov operator behaves essentially like the four-dimensional Laplacian, $\mathcal{M}^{ab} \approx -\partial^2 \delta^{ab}$, exhibiting in this case only positive eigenvalues. This means that the perturbation theory is contained within the Gribov region Ω.

3.3. Some properties of the Gribov region

In equation (29), we defined the region Ω as the set of fields fulfilling the maximal Abelian gauge conditions and for which the operator $\mathcal{M}^{ab}$ is positive definite. Let us now establish some properties of this region. Let us first take a look at the Faddeev–Popov operator $\mathcal{M}^{ab}$, which can be written as

$$\mathcal{M}^{ab} = \mathcal{O}_1^{ab} - \mathcal{O}_2^{ab} - \mathcal{O}_3^{ab}, \tag{30}$$

where

$$\mathcal{O}_1^{ab} = -D_\mu^{ac} D_\mu^{cd}, \qquad \mathcal{O}_2^{ab} = g f^{acd} A_\mu^c D_\mu^{db}, \qquad \mathcal{O}_3^{ab} = g^2 f^{aci} f^{bdi} A_\mu^c A_\mu^d. \qquad (31)$$

As pointed out in [31, 47], the operators $\mathcal{O}_1^{ab}$ and $\mathcal{O}_3^{ab}$ are positive definite. We can easily show this by the following elementary calculation:

$$\langle\psi|\mathcal{O}_1|\psi\rangle = -\int d^4x (\psi^a)^\dagger D_\mu^{ac} D_\mu^{cb} \psi^b = \int d^4x \left(D_\mu^{ac}\psi^c\right)^\dagger D_\mu^{ab}\psi^b = \left\| D_\mu^{ab}\psi^b \right\|^2 \geqslant 0, \qquad (32)$$

$$\langle\psi|\mathcal{O}_3|\psi\rangle = \int d^4x (\psi^a)^\dagger g^2 f^{aci} f^{bdi} A_\mu^c A_\mu^d \psi^b$$
$$= \int d^4x \left(g f^{aci} A_\mu^c \psi^a\right)^\dagger \left(g f^{bdi} A_\mu^d \psi^b\right) = \left\| g f^{abi} A_\mu^b \psi^a \right\|^2 \geqslant 0. \qquad (33)$$

Concerning the operator $\mathcal{O}_2^{ab}$, we note that its trace in color space vanishes, i.e.

$$\operatorname{tr}\mathcal{O}_2 \equiv \mathcal{O}_2^{aa} = g f^{acd} A_\mu^c D_\mu^{da} = g f^{acd} A_\mu^c \left(\delta^{da}\partial_\mu - g f^{dai} A_\mu^i\right)$$
$$= -g f^{aab} A_\mu^b \partial_\mu - g^2 \underbrace{f^{abc} f^{abi}}_{=0} A_\mu^c A_\mu^i = 0. \qquad (34)$$

As a consequence, $\mathcal{O}_2^{ab}$ has both positive and negative eigenvalues. As we shall see in following sections, this feature can be used to establish a few properties of Ω.

3.3.1. The region Ω is unbounded in all diagonal directions. To prove this statement it is sufficient to observe that the purely diagonal field configuration $\left(0, A_\mu^i\right)$ with $A_\mu^i(x)$ transverse, $\partial_\mu A_\mu^i = 0$, fulfills the maximal Abelian gauge condition. Moreover, for this kind of configuration, the Faddeev–Popov operator $\mathcal{M}^{ab}$ reduces to the covariant Laplacian

$$\mathcal{M}^{ab}(0, A^i) = -D_\mu^{ac}(A) D_\mu^{cb}(A), \qquad (35)$$

which is always positive for an arbitrary choice of the transverse diagonal configuration $A_\mu^i(x)$. We see thus that one can freely move along the diagonal directions in field space. The Faddeev–Popov operator will never become negative, meaning that Ω is unbounded in the diagonal directions.

3.3.2. The region Ω is bounded in all off-diagonal directions. To establish this features, we observe that if $\left(B_\mu^a, B_\mu^i\right)$ is a field configuration fulfilling the maximal Abelian conditions, $D_\mu^{ab} B_\mu^b = 0$, $\partial_\mu B_\mu^i = 0$, then the rescaled configuration $\left(\lambda B_\mu^a, B_\mu^i\right)$, where λ is an arbitrary positive constant λ, obeys the same conditions, i.e.

$$D_\mu^{ab}\left(\lambda B_\mu^b\right) = \lambda D_\mu^{ab} B_\mu^b = 0. \qquad (36)$$

Now, let $\left(A_\mu^a, A_\mu^i\right)$ be a configuration which belongs to Ω, namely

$$\langle\psi|\mathcal{M}(A^a, A^i)|\psi\rangle = \langle\psi|\mathcal{O}_1|\psi\rangle - \langle\psi|\mathcal{O}_2|\psi\rangle - \langle\psi|\mathcal{O}_3|\psi\rangle > 0, \qquad (37)$$

and let us evaluate the quantity $\langle\psi|\mathcal{M}|\psi\rangle$ for the rescaled configuration $\left(\lambda A_\mu^a, A_\mu^i\right)$. From expressions (31) one obtains

$$\langle\psi|\mathcal{M}(\lambda A^a, A^i)|\psi\rangle = \langle\psi|\mathcal{O}_1|\psi\rangle - \lambda\langle\psi|\mathcal{O}_2|\psi\rangle - \lambda^2\langle\psi|\mathcal{O}_3|\psi\rangle > 0. \qquad (38)$$

Two cases have to be considered. (i) When $0 \leqslant \lambda \leqslant 1$, the quantity above is always positive since $\langle\psi|\mathcal{O}_1|\psi\rangle > 0$ for every configuration and $\left(A_\mu^a, A_\mu^i\right)$ belongs to the Gribov region by the hypothesis, see equation (37). (ii) When λ is larger than 1, we have to pay

attention to the value of $\langle\psi|\mathcal{O}_2|\psi\rangle$. If this quantity is positive, we just need to take λ as large as we want and certainly a zero mode will be achieved. Note in fact that the third term in equation (38) is always positive, $\langle\psi|\mathcal{O}_3|\psi\rangle > 0$. If $\langle\psi|\mathcal{O}_2|\psi\rangle$ turns out to be negative, it would be sufficient to observe that the contribution coming from $\langle\psi|\mathcal{O}_2|\psi\rangle$ is of first order in λ, while the contribution coming from $\langle\psi|\mathcal{O}_3|\psi\rangle$, which is always positive, is of the order λ^2. Thus, we conclude that even if $\langle\psi|\mathcal{O}_2|\psi\rangle$ is negative, we shall still achieve a zero mode of $\mathcal{M}^{ab}$ for λ sufficiently large. In other words, moving along the off-diagonal directions parameterized by the rescaled configuration $\left(\lambda A_\mu^a, A_\mu^i\right)$, with $\left(A_\mu^a, A_\mu^i\right)$ belonging to the Gribov region Ω, one always encounters a boundary $\partial\Omega$, i.e. the horizon, where the first vanishing eigenvalue of the Faddeev–Popov operator appears. Beyond $\partial\Omega$, the operator $\mathcal{M}^{ab}$ ceases to be positive definite.

Before ending this section, it is worth discussing the case of a particular field configuration[4] which can be handled by making use of the charge conjugation invariance displayed by the maximal Abelian gauge; see [32, 43, 44, 47–49] for a detailed account on this symmetry. This field configuration would correspond to the case in which the eigenvalues of the operators $\mathcal{O}_2$ and $\mathcal{O}_3$ both vanish. If this particular configuration obeys the maximal Abelian gauge conditions (10), (11) and allows for an arbitrary rescaling of the off-diagonal components of the gauge field, then it would give rise to off-diagonal unbounded directions in field space. Luckily, it turns out that this kind of configuration can be excluded by invoking the charge conjugation symmetry. In order to illustrate this issue, let us consider in detail the simpler example of $SU(2)$. In this case, the operator $\mathcal{O}_2$ is absent and the eigenvalues of $\mathcal{O}_3^{ab}(A)$ are given by the characteristic equation

$$\left\| \begin{matrix} g^2 A_\mu^2 A_\mu^2 - \varepsilon_3 & -g^2 A_\mu^1 A_\mu^2 \\ -g^2 A_\mu^1 A_\mu^2 & g^2 A_\mu^1 A_\mu^1 - \varepsilon_3 \end{matrix} \right\| = \left(g^2 A_\mu^2 A_\mu^2 - \varepsilon_3\right)\left(g^2 A_\nu^1 A_\nu^1 - \varepsilon_3\right) - g^4 A_\mu^1 A_\mu^2 A_\nu^1 A_\nu^2 = 0. \tag{39}$$

For configurations of the form

$$A_\mu^1 = \lambda A_\mu^2, \tag{40}$$

with arbitrary real λ, it is easily seen that the characteristic equation displays in fact a zero eigenvalue. Furthermore, this configuration can be made to satisfy the maximal Abelian gauge conditions in a nontrivial way. It would appear then that by choosing λ as large as we wish we could make the Gribov region Ω unbounded along these particular off-diagonal directions. However, it turns out that the Yang–Mills theory in the maximal Abelian gauge possesses a charge conjugation invariance, which is also enjoyed by the horizon term implementing the restriction to the region Ω [32, 43, 44, 47–49]. This symmetry is defined by

$$A_\mu^1 \to A_\mu^1, \qquad A_\mu^2 \to -A_\mu^2, \qquad A_\mu^3 \to -A_\mu^3, \tag{41}$$

and is clearly not obeyed by the configuration (40). As such, these configurations can be excluded from the field space relevant for the Gribov region Ω. Although the general case of $SU(N)$ looks more involved, we expect that a similar reasoning applies as well. Needless to say, the charge conjugation can also be introduced for $SU(N)$.

3.3.3. The convexity of the region Ω.

Let us now discuss the issue of the convexity of the region Ω. Due to the nonlinearity of the gauge conditions, this property will be established for field configurations having the same diagonal components. Namely, consider two configurations $\left(B_\mu^a, A_\mu^i\right)$ and $\left(C_\mu^a, A_\mu^i\right)$ which obey the gauge conditions

$$D_\mu^{ab}(A) B_\mu^b = 0, \qquad D_\mu^{ab}(A) C_\mu^b = 0, \qquad \partial_\mu A_\mu^i = 0, \tag{42}$$

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[4] We are grateful to Nicolas Wschebor for having pointed out to us this possibility.

and belong to the Gribov region, i.e.

$$\mathcal{M}^{ab}(B, A) > 0, \qquad \mathcal{M}^{ab}(C, A) > 0. \tag{43}$$

In order to establish the convexity of the region Ω, we should show that the field configuration $\left(E_\mu^a, A_\mu^i\right)$ defined by

$$E_\mu^a = \alpha B_\mu^a + (1 - \alpha)C_\mu^a, \qquad 0 \leqslant \alpha \leqslant 1, \tag{44}$$

belongs to Ω, i.e. $\mathcal{M}^{ab}(E, A) > 0$. In fact, one can verify straightforwardly that

$$\mathcal{M}^{ab}(E, A) = \alpha \mathcal{M}^{ab}(B, A) + (1 - \alpha)\mathcal{M}^{ab}(C, A) + \alpha(1 - \alpha)\mathcal{O}_3^{ab}(B - C). \tag{45}$$

Since $\mathcal{O}_3^{ab}$ is positive definite, as already mentioned, and since both $\left(B_\mu^a, A_\mu^i\right)$ and $\left(C_\mu^a, A_\mu^i\right)$ belong to Ω, we finally conclude that

$$\mathcal{M}^{ab}(E, A) > 0, \tag{46}$$

showing that $\left(E_\mu^a, A_\mu^i\right)$ belongs to Ω as well. This proves the convexity of Ω.

4. Gribov's statement about field configurations close to the horizon

In [1], Gribov proved, in the case of the Landau gauge, that for any field configuration located within the Gribov region and close to the Gribov horizon, there exists a nearby copy located on the other side of the horizon, i.e. outside of the Gribov region. This result was extended in [32] to the maximal Abelian gauge in $SU(2)$. Here we provide the necessary generalization to $SU(N)$. First, let us consider a field configuration $\left(C_\mu^a, C_\mu^i\right)$ lying on the horizon, that is,

$$D_\mu^{ab}(C)C_\mu^b = 0, \qquad \partial_\mu C_\mu^i = 0 \tag{47}$$

and

$$\mathcal{M}^{ab}(C)\phi_0^b = 0, \tag{48}$$

where $\phi_0^a(x)$ stands for a normalizable zero mode of the Faddeev–Popov operator $\mathcal{M}^{ab}(C)$. For later purposes, it turns out to be useful to introduce the diagonal components $\phi_0^i(x)$ defined according to the second equation of (26) as

$$\phi_0^i = -gf^{abi}\frac{1}{\partial^2}\partial_\mu\left(C_\mu^a\phi_0^b\right). \tag{49}$$

Let $\left(A_\mu^a, A_\mu^i\right)$ be a field configuration belonging to the Gribov region Ω and located close to the horizon $\partial\Omega$. For such a configuration, we can write

$$A_\mu^a = C_\mu^a + a_\mu^a, \qquad A_\mu^i = C_\mu^i + a_\mu^i, \tag{50}$$

where $\left(a_\mu^a, a_\mu^i\right)$ are treated as small perturbative components. As $\left(A_\mu^a, A_\mu^i\right)$ obeys the gauge conditions (10), (11), from equation (47) it follows that

$$D_\mu^{ab}(C)a_\mu^b - gf^{abi}a_\mu^iC_\mu^b = 0, \qquad \partial_\mu a_\mu^i = 0. \tag{51}$$

The eigenvalue of the Faddeev–Popov operator corresponding to the field configuration $\left(A_\mu^a, A_\mu^i\right)$ is easily determined at first order in the small components $\left(a_\mu^a, a_\mu^i\right)$ by using the standard perturbation theory of quantum mechanics, yielding

$$\varepsilon(A) = \varepsilon(C) + \int d^4x \, \phi_0^a \big[2gf^{abi}a_\mu^iD_\mu^{bc}(C)\phi_0^c - gf^{acd}a_\mu^cD_\mu^{db}(C)\phi_0^b$$
$$+ g^2f^{acd}f^{dbi}a_\mu^iC_\mu^c\phi_0^b - g^2f^{aci}f^{bdi}\left(C_\mu^ca_\mu^d + a_\mu^cC_\mu^d\right)\phi_0^b \big]. \tag{52}$$

The first term on the rhs of (52) vanishes because $\left(C_\mu^a, C_\mu^i\right)$ lies on the Gribov horizon $\partial\Omega$.

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We can now proceed as in [1, 32] and introduce a third configuration $\left(\widetilde{A}_\mu^a, \widetilde{A}_\mu^i\right)$, still located close to the Gribov horizon, defined by

$$\widetilde{A}_\mu^a = C_\mu^a + \widetilde{a}_\mu^a, \qquad \widetilde{A}_\mu^i = C_\mu^i + \widetilde{a}_\mu^i, \tag{53}$$

where

$$\widetilde{a}_\mu^a = a_\mu^a - \left(D_\mu^{ab}(C)\phi_0^b + gf^{abc}C_\mu^b\phi_0^c + gf^{abi}C_\mu^b\phi_0^i\right),$$
$$\widetilde{a}_\mu^i = a_\mu^i - \left(\partial_\mu\phi_0^i + gf^{abi}C_\mu^a\phi_0^b\right) \tag{54}$$

are small as compared to $\left(C_\mu^a, C_\mu^i\right)$. It is easy to verify that this new field configuration was constructed in such a way that it fulfills the maximal Abelian gauge conditions

$$D_\mu^{ab}(\widetilde{A})\widetilde{A}_\mu^b = 0, \qquad \partial_\mu\widetilde{A}_\mu^i = 0. \tag{55}$$

Therefore, we can ask ourselves if $\left(\widetilde{A}_\mu^a, \widetilde{A}_\mu^i\right)$ is a Gribov copy of $\left(A_\mu^a, A_\mu^i\right)$, i.e. if these two field configurations can be connected by a gauge transformation. In order to answer this question we shall repeat the argument of [1, 32], amounting to construct such a gauge transformation in an iterative way. Let us suppose thus that it is possible to connect the two configurations by a gauge transformation, so that

$$\begin{aligned}
\widetilde{A}_\mu^a = A_\mu^a &- \left(D_\mu^{ab}\omega^b + gf^{abc}A_\mu^b\omega^c + gf^{abi}A_\mu^b\omega^i\right) \\
&- \frac{g}{2}f^{abc}\omega^b\left(D_\mu^{cd}\omega^d + gf^{cde}A_\mu^d\omega^e + gf^{cdi}A_\mu^d\omega^i\right) \\
&- \frac{g}{2}f^{abi}\omega^b\left(\partial_\mu\omega^i + gf^{cdi}A_\mu^c\omega^d\right) \\
&+ \frac{g}{2}f^{abi}\omega^i\left(D_\mu^{bc}\omega^c + gf^{bcd}A_\mu^c\omega^d + gf^{bcj}A_\mu^c\omega^j\right) + O(\omega^3),
\end{aligned}$$
$$\begin{aligned}
\widetilde{A}_\mu^i = A_\mu^i &- \left(\partial_\mu\omega^i + gf^{abi}A_\mu^a\omega^b\right) \\
&- \frac{g}{2}f^{abi}\omega^a\left(D_\mu^{bc}\omega^c + gf^{bcd}A_\mu^c\omega^d + gf^{bcj}A_\mu^c\omega^j\right) + O(\omega^3), \tag{56}
\end{aligned}$$

where, according to [1], the gauge transformation has been considered till the second order[5] in the gauge parameters ω^i, ω^a. From the maximal Abelian conditions, it follows that

$$\begin{aligned}
\mathcal{M}^{ab}(A)\omega^b + D_\mu^{ab}(A)\Bigg[&-\frac{g}{2}f^{bcd}\omega^c\left(D_\mu^{de}\omega^e + gf^{def}A_\mu^e\omega^f + gf^{dei}A_\mu^e\omega^i\right) \\
&- \frac{g}{2}f^{bci}\omega^c\left(\partial_\mu\omega^i + gf^{dei}A_\mu^d c^e\right) \\
&+ \frac{g}{2}f^{bci}\omega^i\left(D_\mu^{cd}\omega^d + gf^{cde}A_\mu^d\omega^e + gf^{cdj}A_\mu^d\omega^j\right)\Bigg] \\
&+ \frac{g^2}{2}f^{abi}f^{cdi}A_\mu^b\omega^c\left(D_\mu^{de}\omega^e + gf^{def}A_\mu^e\omega^f + gf^{dej}A_\mu^e\omega^j\right) \\
&- gf^{abi}\left(\partial_\mu\omega^i + gf^{cdi}A_\mu^c\omega^d\right)\left(D_\mu^{be}\omega^e + gf^{bef}A_\mu^e\omega^f + gf^{bej}A_\mu^e\omega^j\right) = 0, \tag{57}
\end{aligned}$$

$$\partial_\mu\left[\partial_\mu\omega^i + gf^{abi}A_\mu^a\omega^b + \frac{g}{2}f^{abi}\omega^a\left(D_\mu^{bc}\omega^c + gf^{bcd}A_\mu^c\omega^d + gf^{bcj}A_\mu^c\omega^j\right)\right] = 0. \tag{58}$$

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[5] Note that if we had taken only the first-order gauge transformations, we would obtain

$$\widetilde{A}_\mu^a = A_\mu^a - (D_\mu^{ab}\omega^b + gf^{abc}A_\mu^b\omega^c + gf^{abi}A_\mu^b\omega^i), \qquad \widetilde{A}_\mu^i = A_\mu^i - (\partial_\mu\omega^i + gf^{abi}A_\mu^a\omega^b).$$

Then, applying the maximal Abelian gauge conditions we would obtain the following equations:

$$\mathcal{M}^{ab}(A)\omega^b = 0, \qquad \partial_\mu(\partial_\mu\omega^i + gf^{abi}A_\mu^a\omega^b) = 0,$$

which have no solutions since, by hypothesis, (A_μ^a, A_μ^i) is not located on the horizon $\partial\Omega$.

The next step is to express (ω^a, ω^i) in terms of (ϕ_0^a, ϕ_0^i). Let us start by writing

$$\omega^a = \phi_0^a + \eta_0^a, \qquad \omega^i = \phi_0^i + \eta_0^i, \tag{59}$$

with (η^a, η^i) small when compared to (ϕ_0^a, ϕ_0^i). Equation (57) gives thus the following condition:

$$\partial^2 \eta^a = \mathcal{P}^a(C, a, \phi_0) + \mathcal{Q}^{ab}(C)\eta^b, \tag{60}$$

with $\mathcal{P}^a$ and $\mathcal{Q}^{ab}$ given by

$$\begin{aligned}
\mathcal{P}^a &= 2gf^{abi} a_\mu^i D_\mu^{bc}(C)\phi_0^c - gf^{acd} a_\mu^c D_\mu^{ab}(C)\phi_0^b + g^2 f^{acd} f^{dbi} a_\mu^i C_\mu^c \phi_0^b \\
&\quad - g^2 f^{aci} f^{bdi} \left(C_\mu^c a_\mu^d + a_\mu^c C_\mu^d\right)\phi_0^b + D_\mu^{ab}(C)\Bigg[-\frac{g}{2} f^{bcd} \phi_0^c \left(D_\mu^{de}(C)\phi_0^e\right. \\
&\quad + gf^{def} C_\mu^e \phi_0^f + gf^{dei} C_\mu^e \phi_0^i\right) - \frac{g}{2} f^{bci} \phi_0^c \left(\partial_\mu \phi_0^i + gf^{dei} C_\mu^d \phi_0^e\right) \\
&\quad + \frac{g}{2} f^{bci} \left(D_\mu^{cd}(C)\phi_0^d + gf^{cde} C_\mu^d \phi_0^e + gf^{cdj} C_\mu^d \phi_0^j\right)\Bigg] \\
&\quad + \frac{g^2}{2} f^{abi} f^{cdi} C_\mu^b \phi_0^c \left(D_\mu^{de}(C)\phi_0^e + gf^{def} C_\mu^e \phi_0^f + gf^{dej} C_\mu^e \phi_0^j\right) \\
&\quad - gf^{abi} \left(\partial_\mu \phi_0^i + gf^{cdi} C_\mu^c \phi_0^d\right)\left(D_\mu^{be}(C)\phi_0^e + gf^{bef} C_\mu^e \phi_0^f + gf^{bej} C_\mu^e \phi_0^j\right),
\end{aligned} \tag{61}$$

$$\mathcal{Q}^{ab} = 2gf^{abi} C_\mu^i \partial_\mu - gf^{abc} C_\mu^c D_\mu^{db} - g^2 f^{aci} f^{cbj} C_\mu^i C_\mu^j - g^2 f^{aci} f^{bdi} C_\mu^c C_\mu^d. \tag{62}$$

We notice now that equation (60) can be solved for $\eta^a(x)$ in an iterative way:

$$\eta^a = \frac{1}{\partial^2} \mathcal{P}^a + \frac{1}{\partial^2} \mathcal{Q}^{ab} \frac{1}{\partial^2} \mathcal{P}^b + \cdots, \tag{63}$$

allowing us to obtain a recursive expression for the parameters (ω^a, ω^i) as well as for the gauge transformations (56) in terms of (ϕ_0^a, ϕ_0^i). We have shown thus that the configuration $(\widetilde{A}_\mu^a, \widetilde{A}_\mu^i)$ is indeed a Gribov copy of (A_μ^a, A_μ^i). Furthermore, equation (60) can be used to establish another relevant property. Recalling that $\mathcal{M}^{ab}(C)\phi_0^b = 0$ and that $\mathcal{M}^{ab}$ is Hermitian, we have

$$\int \mathrm{d}^4 x \, \phi_0^a \mathcal{M}^{ab}(C)\eta^b = 0. \tag{64}$$

Therefore, from (60), it follows that

$$\begin{aligned}
\int \mathrm{d}^4 x \, \phi_0^a &\Bigg\{2gf^{abi} a_\mu^i D_\mu^{bc}(C)\phi_0^c - gf^{acd} a_\mu^c D_\mu^{ab}(C)\phi_0^b + g^2 f^{acd} f^{dbi} a_\mu^i C_\mu^c \phi_0^b \\
&\quad - g^2 f^{aci} f^{bdi} \left(C_\mu^c a_\mu^d + a_\mu^c C_\mu^d\right)\phi_0^b + D_\mu^{ab}(C)\Bigg[-\frac{g}{2} f^{bcd} \phi_0^c \left(D_\mu^{de}(C)\phi_0^e\right. \\
&\quad + gf^{def} C_\mu^e \phi_0^f + gf^{dei} C_\mu^e \phi_0^i\right) - \frac{g}{2} f^{bci} \phi_0^c \left(\partial_\mu \phi_0^i + gf^{dei} C_\mu^d \phi_0^e\right) \\
&\quad + \frac{g}{2} f^{bci} \left(D_\mu^{cd}(C)\phi_0^d + gf^{cde} C_\mu^d \phi_0^e + gf^{cdj} C_\mu^d \phi_0^j\right)\Bigg] \\
&\quad + \frac{g^2}{2} f^{abi} f^{cdi} C_\mu^b \phi_0^c \left(D_\mu^{de}(C)\phi_0^e + gf^{def} C_\mu^e \phi_0^f + gf^{dej} C_\mu^e \phi_0^j\right) \\
&\quad - gf^{abi} \left(\partial_\mu \phi_0^i + gf^{cdi} C_\mu^c \phi_0^d\right)\left(D_\mu^{be}(C)\phi_0^e + gf^{bef} C_\mu^e \phi_0^f + gf^{bej} C_\mu^e \phi_0^j\right)\Bigg\} = 0. \tag{65}
\end{aligned}$$

Moreover, the eigenvalue of the Faddeev–Popov operator corresponding to the configuration $\left(\widetilde{A}_{\mu}^{a}, \widetilde{A}_{\mu}^{i}\right)$ can be obtained in the same way as before for $\left(A_{\mu}^{a}, A_{\mu}^{i}\right)$, amounting in fact to replace a_{μ} by $\widetilde{a}_{\mu}$. Thus,

$$\varepsilon(\widetilde{A}) = \int \mathrm{d}^4 x \, \phi_0^a \big[2g f^{abi} \widetilde{a}_{\mu}^i D_{\mu}^{bc}(C) \phi_0^c - g f^{acd} \widetilde{a}_{\mu}^c D_{\mu}^{db}(C) \phi_0^b$$
$$+ g^2 f^{acd} f^{dbi} \widetilde{a}_{\mu}^i C_{\mu}^c \phi_0^b - g^2 f^{aci} f^{bdi} \left(C_{\mu}^c \widetilde{a}_{\mu}^d + \widetilde{a}_{\mu}^c C_{\mu}^d \right) \phi_0^b \big]. \tag{66}$$

Finally, from equations (54), (65) and (66), we obtain

$$\varepsilon(A) = -\varepsilon(\widetilde{A}). \tag{67}$$

Since $\varepsilon(A) > 0$ by hypothesis, it follows that $\varepsilon(\widetilde{A}) = -\varepsilon(A) < 0$, showing that the Gribov copy $\left(\widetilde{A}_{\mu}^{a}, \widetilde{A}_{\mu}^{i}\right)$ is in fact located outside of Ω.

5. Restriction of the domain of integration to the Gribov region Ω

As we have seen before, field configurations located inside Ω and close to the boundary $\partial\Omega$ have copies outside of Ω. Therefore, restricting the domain of integration in the Feynman integral to the Gribov region allows us to get rid of a certain amount of copies[6]. To implement this restriction we shall follow [1, 32] and introduce in the partition function of the theory, equation (24), the factor $\mathcal{V}(\Omega)$ which formally constrains the domain of integration to the region Ω:

$$\mathcal{Z} = \int_{\Omega} \mathrm{d}\mu \, \mathrm{e}^{-S_{\mathrm{YM}}} = \int \mathrm{d}\mu \, \mathrm{e}^{-S_{\mathrm{YM}}} \, \mathcal{V}(\Omega), \tag{68}$$

where $\mathrm{d}\mu$ is the Faddeev–Popov functional measure given in equation (24):

$$\mathrm{d}\mu = \mathcal{D}A^{\mathrm{off}} \mathcal{D}A^{\mathrm{diag}} \, \delta(D \cdot A^{\mathrm{off}}) \delta(\partial \cdot A^{\mathrm{diag}}) (\det \mathcal{M}). \tag{69}$$

In order to obtain an explicit expression for $\mathcal{V}(\Omega)$, we make use of the so-called *no-pole condition* [1], which is a condition on the connected two-point function of the off-diagonal Faddeev–Popov ghost fields. As pointed out in [1], see also [50] for a pedagogical introduction, the no-pole condition stems from the positivity of the Faddeev–Popov operator, $\mathcal{M}^{ab} > 0$, within the Gribov region Ω, according to equation (29). As a consequence, within Ω, the operator $\mathcal{M}^{ab}$ is invertible. Moreover, its inverse $(\mathcal{M}^{ab})^{-1}$ which is nothing but the connected two-point off-diagonal ghost function evaluated in an external gauge field background

$$(\mathcal{M}^{ab}(x, y; A))^{-1} = \frac{\int \mathcal{D}c^{\mathrm{off}} \mathcal{D}\bar{c}^{\mathrm{off}} \, \bar{c}^a(x) c^a(y) \, \mathrm{e}^{\int \mathrm{d}^4 z \, \bar{c}^b(z) \mathcal{M}^{bc}(z) \, c^c(z)}}{\int \mathcal{D}c^{\mathrm{off}} \mathcal{D}\bar{c}^{\mathrm{off}} \, \mathrm{e}^{\int \mathrm{d}^4 z \, \bar{c}^b(z) \mathcal{M}^{bc}(z) \, c^c(z)}} \tag{70}$$

is positive as well within Ω, i.e. $(\mathcal{M}^{ab})^{-1} > 0$. Following [1], the next step to achieve the factor $\mathcal{V}(\Omega)$ is that of considering the quantity

$$\mathcal{G}(k; A) = \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \mathcal{G}(x, y; A) \, \mathrm{e}^{ik(x-y)}, \tag{71}$$

where $\mathcal{G}(x, y; A)$ stands for the trace, in color space, of $(\mathcal{M}^{ab})^{-1}$, namely

$$\mathcal{G}(x, y; A) = \frac{\mathrm{Tr}(\mathcal{M}^{ab}(x, y; A))^{-1}}{N(N-1)}$$
$$= \frac{1}{N(N-1)} \frac{\int \mathcal{D}c^{\mathrm{off}} \mathcal{D}\bar{c}^{\mathrm{off}} \, \bar{c}^a(x) c^a(y) \, \mathrm{e}^{\int \mathrm{d}^4 z \, \bar{c}^b(z) \mathcal{M}^{bc}(z) \, c^c(z)}}{\int \mathcal{D}c^{\mathrm{off}} \mathcal{D}\bar{c}^{\mathrm{off}} \, \mathrm{e}^{\int \mathrm{d}^4 z \, \bar{c}^b(z) \mathcal{M}^{bc}(z) \, c^c(z)}}. \tag{72}$$

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[6] Nowadays, it is known that the restriction to Ω does not enable us to get rid of all copies. As already remarked, a further restriction to a smaller region, known as the fundamental modular region, should be implemented [2]. Though, till now, this seems to be beyond our present capabilities.

Parametrizing thus $\mathcal{G}(k; A)$ as in [1, 32]

$$\mathcal{G}(k; A) \approx \frac{1}{k^2} \frac{1}{1 - \sigma(k; A)} + \frac{\mathcal{J}}{k^4}, \tag{73}$$

where $\mathcal{J}$ is constant, the no-pole condition can be stated as [1]

$$\sigma(k; A) < 1. \tag{74}$$

From this condition, it follows that the off-diagonal ghost propagator has no poles for any finite value of the momentum k, so it cannot change sign by varying k. It will always remain positive, meaning that the Gribov region Ω will never be crossed. The only allowed pole is at $k = 0$, where expression (73) becomes singular, meaning that we are approaching the boundary $\partial\Omega$ where $(\mathcal{M}^{ab})^{-1}$ is in fact singular, due to the zero modes of $\mathcal{M}^{ab}$. Moreover, taking into account that $\sigma(k; A)$ is a decreasing function of the momentum k [1], for the final form of the no-pole condition we might take [1]

$$\sigma(0; A) < 1, \tag{75}$$

which is very suitable for explicit calculations. Therefore, the factor $\mathcal{V}(\Omega)$ implementing the restriction of the domain of integration in the functional integral to the region Ω is

$$\mathcal{V}(\Omega) = \theta(1 - \sigma(0; A)), \tag{76}$$

where θ stands for the step function. Consequently, for the partition function we can write

$$\mathcal{Z} = \int_\Omega d\mu \, e^{-S_{YM}} = \int d\mu \, e^{-S_{YM}} \theta(1 - \sigma(0, A)). \tag{77}$$

5.1. Evaluation of $\mathcal{V}(\Omega)$

Let us face now the characterization of the factor $\mathcal{V}(\Omega)$. To that purpose, we start with expression (72) and evaluate $\mathcal{G}(x, y; A)$ order-by-order in the perturbation theory. Performing the calculation until the third order[7], we obtain[8]

$$
\begin{aligned}
\mathcal{G}(x, y; A) = &\,\mathcal{G}_0(x - y) - g^2 \int d^4z \big[N A^i_\mu(z) A^i_\mu(z) - A^a_\mu(z) A^a_\mu(z) \big] \mathcal{G}_0(z - y) \mathcal{G}_0(x - z) \\
&- g^2 \int d^4z_1 \, d^4z_2 \left[4N A^i_\mu(z_1) A^i_\nu(z_2) + (N - 2) A^a_\mu(z_1) A^a_\nu(z_2) \right] \\
&\times \partial^{z_1}_\mu \mathcal{G}_0(x - z_1) \, \mathcal{G}_0(z_2 - y) \, \partial^{z_2}_\nu \mathcal{G}_0(z_1 - z_2) \\
&+ \frac{g^3}{N(N - 1)} \int d^4z_1 \, d^4z_2 \, f^{abc} f^{bde} f^{adi} A^c_\mu(z_1) A^e_\nu(z_2) A^i_\nu(z_2) \big[\big(\partial^{z_1}_\mu \mathcal{G}_0(x - z_1) \big) \\
&\times \mathcal{G}_0(z_2 - y) \mathcal{G}_0(z_1 - z_2) + \mathcal{G}_0(x - z_2) \, \mathcal{G}_0(z_1 - y) \, \partial^{z_1}_\mu \mathcal{G}_0(z_2 - z_1) \big] \\
&- \frac{g^3}{N(N - 1)} \int d^4z_1 \, d^4z_2 \, d^4z_3 \big(8 f^{abi} f^{bcj} f^{cak} A^i_\mu(z_1) A^j_\nu(z_2) A^k_\sigma(z_3) \\
&+ 4 f^{abi} f^{bdj} f^{dac} A^i_\mu(z_1) A^j_\nu(z_2) A^c_\sigma(z_3) + 4 f^{abi} f^{bdc} f^{daj} A^i_\mu(z_1) A^c_\nu(z_2) A^j_\sigma(z_3) \\
&+ 4 f^{abc} f^{bdi} f^{daj} A^c_\mu(z_1) A^i_\nu(z_2) A^j_\sigma(z_3) + 2 f^{abi} f^{bdc} f^{dae} A^i_\mu(z_1) A^c_\nu(z_2) A^e_\sigma(z_3) \\
&+ 2 f^{abc} f^{bdi} f^{dae} A^c_\mu(z_1) A^i_\nu(z_2) A^e_\sigma(z_3) + 2 f^{abc} f^{bde} f^{dai} A^c_\mu(z_1) A^e_\nu(z_2) A^i_\sigma(z_3) \\
&+ f^{abc} f^{bde} f^{daf} A^c_\mu(z_1) A^e_\nu(z_2) A^f_\sigma(z_3) \big) \\
&\times \big[\big(\partial^{z_1}_\mu \mathcal{G}_0(x - z_1) \big) \mathcal{G}_0(z_2 - y) \big(\partial^{z_3}_\sigma \mathcal{G}_0(z_1 - z_3) \big) \partial^{z_2}_\nu \mathcal{G}_0(z_3 - z_2) \big], \tag{78}
\end{aligned}
$$

---

[7] In [1, 32] the calculations were performed only till the second order, as this is sufficient to determine how the restriction to Ω affects the form of the tree-level propagators. Furthermore, a third-order calculation will be needed in order to characterize the horizon function, as recently done in [51] in the case of the Landau gauge.

[8] More details of this lengthy calculation can be found in the appendix.

where

$$\mathcal{G}_0(x - y) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2} \, \mathrm{e}^{-\mathrm{i}q(x-y)}. \tag{79}$$

In momentum space we can write

$$\begin{aligned}
\mathcal{G}(k; A) &= \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \mathcal{G}(x, y; A) \, \mathrm{e}^{\mathrm{i}k(x-y)} \\
&= \frac{1}{k^2} - \frac{g^2}{k^4} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left(\frac{A_\mu^i(q) A_\mu^i(-q)}{N - 1} - \frac{A_\mu^a(q) A_\mu^a(-q)}{N(N - 1)} \right) \\
&\quad + g^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{k_\mu(k - q)_\nu}{k^4(k - q)^2} \left(\frac{4}{(N - 1)} A_\mu^i(q) A_\nu^i(-q) + \frac{N - 2}{N(N - 1)} A_\mu^a(q) A_\nu^a(-q) \right) \\
&\quad + \frac{g^3}{N(N - 1)} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} \left(\frac{\mathrm{i}k_\mu}{k^4(p + k)^2} + \frac{\mathrm{i}(k - p)_\mu}{k^4(k - p)^2} \right) f^{abc} f^{bde} f^{adi} A_\mu^c(-p) \\
&\quad \times A_\nu^e(-q) A_\nu^i(p + q) + \frac{g^3}{N(N - 1)} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{i}k_\sigma(k - p)_\mu(k - p - q)_\nu}{k^4(k - p)^2(k - p - q)^2} \\
&\quad \times \big(8 f^{abi} f^{bcj} f^{cak} A_\mu^i(-p) A_\nu^j(-q) A_\sigma^k(p + q) \\
&\quad + 4 f^{abi} f^{bdj} f^{dac} A_\mu^i(-p) A_\nu^j(-q) A_\sigma^c(p + q) \\
&\quad + 4 f^{abi} f^{bdc} f^{daj} A_\mu^i(-p) A_\nu^c(-q) A_\sigma^j(p + q) \\
&\quad + 4 f^{abc} f^{bdi} f^{daj} A_\mu^c(-p) A_\nu^i(-q) A_\sigma^j(p + q) \\
&\quad + 2 f^{abi} f^{bdc} f^{dae} A_\mu^i(-p) A_\nu^c(-q) A_\sigma^e(p + q) \\
&\quad + 2 f^{abc} f^{bdi} f^{dae} A_\mu^c(-p) A_\nu^i(-q) A_\sigma^e(p + q) \\
&\quad + 2 f^{abc} f^{bde} f^{dai} A_\mu^c(-p) A_\nu^e(-q) A_\sigma^i(p + q) \\
&\quad + f^{abc} f^{bde} f^{daf} A_\mu^c(-p) A_\nu^e(-q) A_\sigma^f(p + q) \big). \tag{80}
\end{aligned}$$

According to equation (73), the expression above can be rewritten as

$$\mathcal{G}(k; A) = \frac{1}{k^2}(1 + \sigma(k; A)) + \frac{\mathcal{J}}{k^4}, \tag{81}$$

where the form factor $\sigma(k; A)$ and $\mathcal{J}$ are given by

$$\begin{aligned}
\sigma(k; A) &= g^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{k_\mu(k - q)_\nu}{k^2(k - q)^2} \left(\frac{4}{(N - 1)} A_\mu^i(q) A_\nu^i(-q) + \frac{N - 2}{N(N - 1)} A_\mu^a(q) A_\nu^a(-q) \right) \\
&\quad + \frac{g^3}{N(N - 1)} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} \left(\frac{\mathrm{i}k_\mu}{k^2(p + k)^2} + \frac{\mathrm{i}(k - p)_\mu}{k^2(k - p)^2} \right) \\
&\quad \times f^{abc} f^{bde} f^{adi} A_\mu^c(-p) A_\nu^e(-q) A_\nu^i(p + q) \\
&\quad + \frac{g^3}{N(N - 1)} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{i}k_\sigma(k - p)_\mu(k - p - q)_\nu}{k^2(k - p)^2(k - p - q)^2} \\
&\quad \times \big(8 f^{abi} f^{bcj} f^{cak} A_\mu^i(-p) A_\nu^j(-q) A_\sigma^k(p + q) \\
&\quad + 4 f^{abi} f^{bdj} f^{dac} A_\mu^i(-p) A_\nu^j(-q) A_\sigma^c(p + q) \\
&\quad + 4 f^{abi} f^{bdc} f^{daj} A_\mu^i(-p) A_\nu^c(-q) A_\sigma^j(p + q) \\
&\quad + 4 f^{abc} f^{bdi} f^{daj} A_\mu^c(-p) A_\nu^i(-q) A_\sigma^j(p + q) \\
&\quad + 2 f^{abi} f^{bdc} f^{dae} A_\mu^i(-p) A_\nu^c(-q) A_\sigma^e(p + q) \\
&\quad + 2 f^{abc} f^{bdi} f^{dae} A_\mu^c(-p) A_\nu^i(-q) A_\sigma^e(p + q)
\end{aligned}$$

$$+ 2 f^{abc} f^{bde} f^{dai} A_\mu^c(-p) A_\nu^e(-q) A_\sigma^i(p+q)$$
$$+ f^{abc} f^{bde} f^{daf} A_\mu^c(-p) A_\nu^e(-q) A_\sigma^f(p+q)), \tag{82}$$

$$\mathcal{J} = -g^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left(\frac{A_\mu^i(q) A_\mu^i(-q)}{N-1} - \frac{A_\mu^a(q) A_\mu^a(-q)}{N(N-1)} \right). \tag{83}$$

Note that $\mathcal{J}$ is independent of the external momentum k. Also, expression (82) can be simplified by making use of the maximal Abelian gauge conditions in momentum space (13), namely

$$q_\mu A_\mu^i = 0, \qquad q_\mu A_\mu^a(q) = \mathrm{i} g f^{abi} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} A_\mu^i(q-p) A_\mu^b(p). \tag{84}$$

Then, for $\sigma(k; A)$ we obtain

$$\sigma(k; A) = g^2 \frac{k_\mu k_\nu}{k^2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(k-q)^2} \left(\frac{4}{(N-1)} A_\mu^i(q) A_\nu^i(-q) + \frac{N-2}{N(N-1)} A_\mu^a(q) A_\nu^a(-q) \right)$$

$$+ \frac{\mathrm{i} g^3 (N-2)}{N(N-1)} \frac{k_\mu}{k^2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(k-q)^2} f^{abi} A_\mu^a(q) A_\nu^i(p-q) A_\nu^b(-p)$$

$$+ \frac{\mathrm{i} g^3}{N(N-1)} \frac{k_\mu}{k^2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} \left(\frac{1}{(p+k)^2} + \frac{1}{(k-p)^2} \right)$$

$$\times f^{abc} f^{bde} f^{adi} A_\mu^c(-p) A_\nu^e(-q) A_\nu^i(p+q)$$

$$+ \frac{\mathrm{i} g^3}{N(N-1)} \frac{k_\sigma k_\mu}{k^2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{(k-p)_\nu}{(k-p)^2 (k-p-q)^2}$$

$$\times \big(8 f^{abi} f^{bcj} f^{cak} A_\mu^i(-p) A_\nu^j(-q) A_\sigma^k(p+q)$$

$$+ 4 f^{abi} f^{bdj} f^{dac} A_\mu^i(-p) A_\nu^j(-q) A_\sigma^c(p+q)$$

$$+ 4 f^{abi} f^{bdc} f^{daj} A_\mu^i(-p) A_\nu^c(-q) A_\sigma^j(p+q)$$

$$+ 4 f^{abc} f^{bdi} f^{daj} A_\mu^c(-p) A_\nu^i(-q) A_\sigma^j(p+q)$$

$$+ 2 f^{abi} f^{bdc} f^{dae} A_\mu^i(-p) A_\nu^c(-q) A_\sigma^e(p+q)$$

$$+ 2 f^{abc} f^{bdi} f^{dae} A_\mu^c(-p) A_\nu^i(-q) A_\sigma^e(p+q)$$

$$+ 2 f^{abc} f^{bde} f^{dai} A_\mu^c(-p) A_\nu^e(-q) A_\sigma^i(p+q)$$

$$+ f^{abc} f^{bde} f^{daf} A_\mu^c(-p) A_\nu^e(-q) A_\sigma^f(p+q) \big) + O(g^4). \tag{85}$$

Using the relation

$$f^{abc} f^{bde} f^{adi} = -\frac{N-2}{2} f^{cei}, \tag{86}$$

and performing the change of variables $p \to -q$ and $q \to p$ in the third line on the rhs of expression (85), we obtain

$$\sigma(k; A) = g^2 \frac{k_\mu k_\nu}{k^2} I_{\mu\nu}^{(1)}(k) + \frac{\mathrm{i} g^3 (N-2)}{2N(N-1)} \frac{k_\mu}{k^2} I_\mu^{(2)}(k)$$

$$+ \frac{\mathrm{i} g^3}{N(N-1)} \left(\frac{k_\mu k_\nu k_\sigma}{k^2} I_{\mu\nu\sigma}^{(3)}(k) - \frac{k_\mu k_\sigma}{k^2} I_{\mu\sigma}^{(4)}(k) \right) + O(g^4), \tag{87}$$

where

$$I_{\mu\nu}^{(1)}(k) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(k-q)^2} \left(\frac{4}{(N-1)} A_\mu^i(q) A_\nu^i(-q) + \frac{N-2}{N(N-1)} A_\mu^a(q) A_\nu^a(-q) \right), \tag{88}$$

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$$I_\mu^{(2)}(k) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} \left(\frac{1}{(k-q)^2} - \frac{1}{(k+q)^2} \right) f^{abi} A_\mu^a(q) A_\nu^i(p-q) A_\nu^b(-p), \qquad (89)$$

$$\begin{aligned}
I_{\mu\nu\sigma}^{(3)}(k) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(k-p)^2(k-p-q)^2} \big(& 8 f^{abi} f^{bcj} f^{cak} A_\mu^i(-p) A_\nu^j(-q) A_\sigma^k(p+q) \\
& + 4 f^{abi} f^{bdj} f^{dac} A_\mu^i(-p) A_\nu^j(-q) A_\sigma^c(p+q) \\
& + 4 f^{abi} f^{bdc} f^{daj} A_\mu^i(-p) A_\nu^c(-q) A_\sigma^j(p+q) \\
& + 4 f^{abc} f^{bdi} f^{daj} A_\mu^c(-p) A_\nu^i(-q) A_\sigma^j(p+q) \\
& + 2 f^{abi} f^{bdc} f^{dae} A_\mu^i(-p) A_\nu^c(-q) A_\sigma^e(p+q) \\
& + 2 f^{abc} f^{bdi} f^{dae} A_\mu^c(-p) A_\nu^i(-q) A_\sigma^e(p+q) \\
& + 2 f^{abc} f^{bde} f^{dai} A_\mu^c(-p) A_\nu^e(-q) A_\sigma^i(p+q) \\
& + f^{abc} f^{bde} f^{daf} A_\mu^c(-p) A_\nu^e(-q) A_\sigma^f(p+q) \big),
\end{aligned} \qquad (90)$$

$$\begin{aligned}
I_{\mu\sigma}^{(4)}(k) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{p_\nu}{(k-p)^2(k-p-q)^2} \big(& 8 f^{abi} f^{bcj} f^{cak} A_\mu^i(-p) A_\nu^j(-q) A_\sigma^k(p+q) \\
& + 4 f^{abi} f^{bdj} f^{dac} A_\mu^i(-p) A_\nu^j(-q) A_\sigma^c(p+q) \\
& + 4 f^{abi} f^{bdc} f^{daj} A_\mu^i(-p) A_\nu^c(-q) A_\sigma^j(p+q) \\
& + 4 f^{abc} f^{bdi} f^{daj} A_\mu^c(-p) A_\nu^i(-q) A_\sigma^j(p+q) \\
& + 2 f^{abi} f^{bdc} f^{dae} A_\mu^i(-p) A_\nu^c(-q) A_\sigma^e(p+q) \\
& + 2 f^{abc} f^{bdi} f^{dae} A_\mu^c(-p) A_\nu^i(-q) A_\sigma^e(p+q) \\
& + 2 f^{abc} f^{bde} f^{dai} A_\mu^c(-p) A_\nu^e(-q) A_\sigma^i(p+q) \\
& + f^{abc} f^{bde} f^{daf} A_\mu^c(-p) A_\nu^e(-q) A_\sigma^f(p+q) \big).
\end{aligned} \qquad (91)$$

In order to be able to evaluate the form factor $\sigma(k; A)$ at zero momentum, equation (75),

$$\sigma(0; A) \equiv \lim_{k \to 0} \sigma(k; A), \qquad (92)$$

we need to analyze the limit $k \to 0$ of the integrals (88)–(91). As these integrals depend explicitly on the Fourier components of the gauge fields, $A_\mu^i(q)$, $A_\mu^a(q)$, the existence of the zero momentum limit, $k \to 0$, of expressions (88)–(91) relies on the infrared behavior of the fields $A_\mu^i(q)$ and $A_\mu^a(q)$. For example, considering (88), it is apparent that the zero momentum integral

$$I_{\mu\nu}^{(1)}(0) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2} \left(\frac{4}{(N-1)} A_\mu^i(q) A_\nu^i(-q) + \frac{N-2}{N(N-1)} A_\mu^a(q) A_\nu^a(-q) \right) \qquad (93)$$

does exist provided the quantities $A_\mu^i(q) A_\mu^i(-q)$, $A_\mu^a(q) A_\mu^a(-q)$ are not singular at $q \approx 0$, i.e.

$$\begin{aligned}
A_\mu^i(q) A_\mu^i(-q)\big|_{q^2 \approx 0} &\approx (q^2)^\alpha, \qquad \alpha > -2, \\
A_\mu^a(q) A_\mu^a(-q)\big|_{q^2 \approx 0} &\approx (q^2)^\beta, \qquad \beta > -2,
\end{aligned} \qquad (94)$$

so that expression (93) is integrable at $q \approx 0$. At the present stage, a formal proof of the infrared behavior expressed by (94) cannot be given. An explicit check of the validity of (94) can be done only after the evaluation of the gluon propagators. In other words, as done in the original paper by Gribov [1], (94) is assumed as the working hypothesis in order to ensure the existence of the zero momentum limit of expressions (88)–(91). Afterward, this

assumption has to be checked out by looking at the behavior of the diagonal and off-diagonal gluon propagators at the origin. Nevertheless, we can provide a physical justification for assuming (94), which follows from the implementation of the restriction to the Gribov region Ω. As also observed in [1] in the case of the Landau gauge, the restriction to the region Ω amounts to put a boundary in field space, namely the Gribov horizon, which manifests itself through the appearance of a dynamical mass parameter γ, which is known as the Gribov mass [1]. As we shall see in the next sections, such a parameter will provide a natural infrared scale for the field configurations, resulting in a gluon propagator which is not singular at the origin, thus justifying the starting working hypothesis (94). In fact, looking at the expressions for the diagonal and off-diagonal propagators given in equations (209), (210), one sees that they are not singular at the origin.

The integral $I_\mu^{(2)}(k)$ is automatically vanishing when $k = 0$; the integrals $I_{\mu\nu}^{(1)}(0)$ and $I_{\mu\nu}^{(4)}(0)$ must be proportional to $\delta_{\mu\nu}$ in the limit $k \to 0$, since $\delta_{\mu\nu}$ is the only Lorentz invariant second-rank tensor available; and finally $I_{\mu\nu\sigma}^{(3)}(k)$ must vanish when $k \to 0$, as in four dimensions there is no Lorentz invariant third-order rank tensor. Summarizing, we have

$$
\begin{aligned}
I_{\mu\nu}^{(1)}(0) &= \frac{I_{\lambda\lambda}^{(1)}(0)}{4}\,\delta_{\mu\nu} &&\text{(by Lorentz covariance)},\\[4pt]
I_{\mu}^{(2)}(0) &= 0 &&\text{(automatically)},\\[4pt]
I_{\mu\nu\sigma}^{(3)}(0) &= 0 &&\text{(by Lorentz covariance)},\\[4pt]
I_{\mu\nu}^{(4)}(0) &= \frac{I_{\lambda\lambda}^{(4)}(0)}{4}\,\delta_{\mu\nu} &&\text{(by Lorentz covariance)}.
\end{aligned}
\tag{95}
$$

Thus, taking the limit $k \to 0$ of $\sigma(k; A)$ for the form factor $\sigma(0; A)$, we obtain

$$
\begin{aligned}
\sigma(0; A) &\equiv \lim_{k\to 0}\sigma(k; A)\\[6pt]
&= g^2\frac{I_{\lambda\lambda}^{(1)}(0)}{4} - \frac{ig^3}{N(N-1)}\frac{I_{\lambda\lambda}^{(4)}(0)}{4} + O(g^4)\\[6pt]
&= g^2\int\frac{\mathrm{d}^4q}{(2\pi)^4}\left(\frac{1}{(N-1)}\frac{A_\lambda^i(q)A_\lambda^i(-q)}{q^2} + \frac{N-2}{4N(N-1)}\frac{A_\lambda^a(q)A_\lambda^a(-q)}{q^2}\right)\\[6pt]
&\quad - \frac{ig^3}{N(N-1)}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{p_\mu}{p^2(p+q)^2}\\[6pt]
&\quad \times \Bigg(2f^{abi}f^{bcj}f^{cak}A_\lambda^i(-p)A_\mu^j(-q)A_\lambda^k(p+q)\\[6pt]
&\quad + f^{abi}f^{bdj}f^{dac}A_\lambda^i(-p)A_\mu^j(-q)A_\lambda^c(p+q)\\[6pt]
&\quad + f^{abi}f^{bdc}f^{daj}A_\lambda^i(-p)A_\mu^c(-q)A_\lambda^j(p+q)\\[6pt]
&\quad + f^{abc}f^{bdi}f^{daj}A_\lambda^c(-p)A_\mu^i(-q)A_\lambda^j(p+q)\\[6pt]
&\quad + \frac{1}{2}f^{abi}f^{bdc}f^{dae}A_\lambda^i(-p)A_\mu^c(-q)A_\lambda^e(p+q)\\[6pt]
&\quad + \frac{1}{2}f^{abc}f^{bdi}f^{dae}A_\lambda^c(-p)A_\mu^i(-q)A_\lambda^e(p+q)\\[6pt]
&\quad + \frac{1}{2}f^{abc}f^{bde}f^{dai}A_\lambda^c(-p)A_\mu^e(-q)A_\lambda^i(p+q)\\[6pt]
&\quad + \frac{1}{4}f^{abc}f^{bde}f^{daf}A_\lambda^c(-p)A_\mu^e(-q)A_\lambda^f(p+q)\Bigg).
\end{aligned}
\tag{96}
$$

As a useful check, we note here that, for $N = 2$, the previous results obtained in [32] are recovered.

5.2. Gribov's quadratic approximation and the gap equation

Having characterized the factor $\sigma(0; A)$, equation (96), we start to analyze the consequences on the theory of the restriction of the domain of integration to the Gribov region, as stated by equation (77). As we shall see, this will amount to modify the starting gauge field theory by the addition of a nonperturbative term, usually referred to as the horizon function [7, 8, 48]. To illustrate this point, we discuss first the form factor $\sigma(0; A)$ in the so-called quadratic approximation [1], i.e. we shall first consider $\sigma(0; A)$ till the second order in the gauge fields. Thus,

$$\sigma(0; A) = g^2 \frac{1}{V} \sum_q \left(\frac{1}{(N-1)} \frac{A_\lambda^i(q) A_\lambda^i(-q)}{q^2} + \frac{N-2}{4N(N-1)} \frac{A_\lambda^a(q) A_\lambda^a(-q)}{q^2} \right), \tag{97}$$

where, according to [1], a finite volume V has been considered. As in the case of $SU(2)$ [32], the quadratic approximation will enable us to obtain the first term of the horizon function, which will play an important role for its all order extension.

Using the integral representation of the step function,

$$\theta(x) = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\zeta}{2\pi i \beta} e^{\zeta x}, \tag{98}$$

for the partition function (77) we obtain the following expression:

$$\mathcal{Z} = \int \frac{d\zeta}{2\pi i \zeta} d\mu \, e^{-S_{\text{YM}}+\zeta(1-\sigma(0;A))}. \tag{99}$$

Moreover, in the quadratic approximation, $\mathcal{Z} \to \mathcal{Z}_{\text{quad}}$:

$$\mathcal{Z}_{\text{quad}} = \lim_{\substack{\alpha \to 0 \\ \beta \to 0}} \int \frac{d\zeta}{2\pi i \zeta} e^\zeta \int \mathcal{D}A^{\text{off}} \mathcal{D}A^{\text{diag}} e^{-\frac{1}{2}\sum_k (A_\mu^a(k)\mathcal{P}_{\mu\nu}^{ab}(k;\zeta,\alpha)A_\nu^b(-k)+A_\mu^i(k)\mathcal{Q}_{\mu\nu}^{ij}(k;\zeta,\beta)A_\nu^j(-k))},$$

$$\mathcal{P}_{\mu\nu}^{ab}(k; \zeta, \alpha) = \left[\left(k^2 + \frac{g^2(N-2)\zeta}{2N(N-1)Vk^2} \right) \delta_{\mu\nu} - \left(1 - \frac{1}{\alpha} \right) k_\mu k_\nu \right] \delta^{ab}, \tag{100}$$

$$\mathcal{Q}_{\mu\nu}^{ij}(k; \zeta, \beta) = \left[\left(k^2 + \frac{2g^2\zeta}{(N-1)Vk^2} \right) \delta_{\mu\nu} - \left(1 - \frac{1}{\beta} \right) k_\mu k_\nu \right] \delta^{ij},$$

where use has been made of the relations

$$\delta(D \cdot A^{\text{off}}) \propto \lim_{\alpha \to 0} e^{-\frac{1}{2\alpha} \int d^4x (D_\mu^{ab} A_\mu^b)^2},$$

$$\delta(\partial \cdot A^{\text{diag}}) \propto \lim_{\beta \to 0} e^{-\frac{1}{2\beta} \int d^4x (\partial_\mu A_\mu^i)^2},$$

$$A_\mu^{a,i}(x) = \frac{1}{V^{1/2}} \sum_k A_\mu^{a,i}(k) e^{-ikx},$$

$$\delta_{k,k'} = \frac{1}{V} \int_V d^4x \, e^{ix(k-k')}. \tag{101}$$

Integrating over the gauge fields gives

$$\mathcal{Z}_{\text{quad}} = \lim_{\substack{\alpha \to 0 \\ \beta \to 0}} \int \frac{d\zeta}{2\pi i} e^{f(\zeta;\alpha,\beta)},$$

$$f(\zeta; \alpha, \beta) = \zeta - \ln \zeta - \frac{1}{2} \ln \det \mathcal{P}_{\mu\nu}^{ab}(k; \zeta, \alpha) - \frac{1}{2} \ln \det \mathcal{Q}_{\mu\nu}^{ij}(k; \zeta, \beta)$$

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$$= \zeta - \ln \zeta - \frac{3N(N-1)}{2} \sum_k \left[\ln \left(k^2 + \frac{g^2(N-2)\zeta}{2N(N-1)Vk^2} \right) \right.$$

$$\left. + \frac{1}{3} \ln \left(\frac{k^2}{\alpha} + \frac{g^2(N-2)\zeta}{2N(N-1)Vk^2} \right) \right] - \frac{3(N-1)}{2}$$

$$\times \sum_k \left[\ln \left(k^2 + \frac{2g^2\zeta}{(N-1)Vk^2} \right) + \frac{1}{3} \ln \left(\frac{k^2}{\beta} + \frac{2g^2\zeta}{(N-1)Vk^2} \right) \right]. \tag{102}$$

The integral over ζ can be evaluated in a saddle point approximation [1]

$$\mathcal{Z}_{\text{quad}} \approx \lim_{\substack{\alpha \to 0 \\ \beta \to 0}} e^{f(\zeta^\star; \alpha, \beta)}, \qquad \left. \frac{df(\zeta)}{d\zeta} \right|_{\zeta = \zeta^\star} = 0. \tag{103}$$

Condition (103) gives us the following relation[9]:

$$1 - \frac{1}{\zeta^\star} - \frac{1}{V} \sum_k \frac{3g^2}{k^4 + \frac{2g^2\zeta^\star}{(N-1)V}} - \frac{1}{V} \sum_k \frac{3(N-2)g^2/4}{k^4 + \frac{(N-2)g^2\zeta^\star}{2N(N-1)V}} = 0. \tag{104}$$

As in [1], in order to take the thermodynamic limit $V \to \infty$, we introduce the so-called Gribov parameter γ:

$$\gamma^4 = \lim_{V \to \infty} \frac{\zeta^\star}{2V}. \tag{105}$$

Therefore, in the infinite volume limit $V \to \infty$, we obtain

$$\int \frac{d^4k}{(2\pi)^4} \frac{3g^2}{k^4 + \frac{4g^2\gamma^4}{N-1}} + \int \frac{d^4k}{(2\pi)^4} \frac{3(N-2)g^2/4}{k^4 + \frac{(N-2)g^2\gamma^4}{N(N-1)}} = 1, \tag{106}$$

where we have neglected the term $1/\zeta^\star$. As observed in [1], the Gribov parameter is not an independent parameter of the theory. It turns out to be determined at the quantum level in a self-consistent way as a function of the coupling constant. In fact, equation (106) is nothing but the gap equation enabling us to express γ as a function of the coupling constant g and of the invariant scale Λ_{QCD}, see [13].

Coming back to the partition function in the saddle point approximation, taking the form factor $\sigma(0; k)$ in the quadratic approximation and performing the thermodynamic limit, we can write

$$\mathcal{Z} = \mathcal{N}' \int d\mu \, e^{-(S_{\text{YM}} + S_{\text{G}})}. \tag{107}$$

Here, $\mathcal{N}'$ is an irrelevant normalization factor and S_{G} is a nonlocal term, given by

$$S_{\text{G}} = \lim_{V \to \infty} \zeta^\star \sigma(0; A)$$

$$= g^2\gamma^4 \int d^4x \, d^4y \left(\frac{2}{N-1} A^i_\mu(x) \mathcal{G}_0(x-y) A^i_\mu(y) + \frac{N-2}{2N(N-1)} A^a_\mu(x) \mathcal{G}_0(x-y) A^a_\mu(y) \right)$$

$$= g^2\gamma^4 \int d^4x \left(\frac{2}{N-1} A^i_\mu \left(\frac{1}{-\partial^2} \right) A^i_\mu + \frac{N-2}{2N(N-1)} A^a_\mu \left(\frac{1}{-\partial^2} \right) A^a_\mu \right). \tag{108}$$

Equation (107) shows that the implementation of the restriction of the domain of integration to the Gribov region Ω can be achieved by adding to the Yang–Mills action a nonlocal term given in equation (108). Expression (108) represents only the first term of what is usually called the horizon function, whose final expression will be discussed in details in the next section.

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[9] Note that the gauge parameters α, β have been set to zero, see equation (101).

6. Characterization of the horizon function

6.1. A few words on the horizon function in the Landau gauge and in $SU(2)$ maximal Abelian gauge

Before entering into the technical details of the determination of the horizon function for $SU(N)$, it is useful to give a short survey of what has been already achieved in other cases. In the Landau gauge, it has been shown by Zwanziger [7, 8] that the horizon function implementing the restriction to the Gribov region Ω[10] is

$$H_{\text{Landau}} = g^2 \int \mathrm{d}^4x \, \mathrm{d}^4y \, D_\mu^{AE}(x)[(-\partial \cdot D)^{-1}]^{AB}(x, y)D_\mu^{BE}(y), \tag{110}$$

where $[(-\partial \cdot D)^{-1}]^{AB}(x, y)$ is the inverse of the Faddeev–Popov operator

$$-(\partial \cdot D)^{AB} = -\partial_\mu D_\mu^{AB} = -\partial_\mu\big(\partial_\mu \delta^{AB} - gf^{ABC}A_\mu^C\big). \tag{111}$$

Accordingly, for the partition function implementing the restriction of the domain of integration to the Gribov region, we write

$$\mathcal{Z}_{\text{Landau}} = \int_\Omega \mathrm{d}\mu \, \mathrm{e}^{-S_{\text{YM}}} = \int \mathrm{d}\mu \, \mathrm{e}^{-(S_{\text{YM}}+\gamma^4 H_{\text{Landau}})}, \tag{112}$$

where, to the first order, the Gribov parameter γ is defined by the gap equation [7, 8]

$$1 = \frac{3Ng^2}{4} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^4 + 2Ng^2\gamma^4}. \tag{113}$$

One should observe that expression (110) is nonlocal. Willing to give a quantum field theory meaning to such an expression, one should be able to cast it into a local and renormalizable form. We mention here that a consistent, local and renormalizable framework for expression (110) is in fact available; see [11] for an updated discussion of this nontrivial issue. Moreover, it is worth to point out that the Zwanziger horizon function (110) turns out to be equivalent to Gribov's ghost form factor, as shown in [51] by comparing the expansion of both expressions till the third in the gauge field.

In the case of the $SU(2)$ Yang–Mills theory in the maximal Abelian gauge, the results found in [48] are very similar to the ones obtained by Zwanziger in the Landau gauge. Summarizing, the form of the horizon function obtained by requiring that it can be cast in local form, it can be renormalizable and it reduces to Gribov's ghost form factor in the quadratic approximation is given by

$$H_{\text{MAG}}^{SU(2)} = g^2 \int \mathrm{d}^4x \, \mathrm{d}^4x \, \varepsilon^{ac} A_\mu(x)(\mathcal{M}^{-1})^{ab}(x, y)\varepsilon^{bc} A_\mu(y), \tag{114}$$

where $(\mathcal{M}^{-1})^{ab}(x, y)$ is the inverse of the Faddeev–Popov operator

$$\mathcal{M}^{ab} = -D_\mu^{ac} D_\mu^{cb} - g^2 \varepsilon^{ac}\varepsilon^{bd} A_\mu^c A_\mu^d. \tag{115}$$

6.2. Candidates for the horizon function in $SU(N)$

Let us face now the problem of finding the horizon term for the general case of $SU(N)$. The first step for accomplishing this task is that of selecting the possible candidates which are

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[10] We recall here that, analogously to the case of the maximal Abelian gauge, the Gribov region in the Landau gauge is defined through the positivity of the Faddeev–Popov operator, i.e.

$$\Omega = \{A_\mu^A; \partial_\mu A_\mu^A = 0; -\partial_\mu(\partial_\mu \delta^{AB} - gf^{ABC}A_\mu^C) > 0\}. \tag{109}$$

compatible with the following requirements:

(1) it should give back to the Gribov term (108) in the quadratic approximation;
(2) it should reproduce the particular case (114) when $N = 2$;
(3) it should be localizable and compatible with the requirement of power counting renormalizability.

Five possible candidates which are compatible with the aforementioned requirements have been found, namely

$$H^{1st} = \frac{2g^2}{N(N-1)} \int_{x,y} f^{aci} A^i_{\mu,x} (\mathcal{M}^{-1})^{ab}_{x,y} f^{bcj} A^j_{\mu,y}$$

$$+ \frac{g^2}{2N(N-1)} \int_{x,y} f^{acd} A^d_{\mu,x} (\mathcal{M}^{-1})^{ab}_{x,y} f^{bce} A^e_{\mu,y} \tag{116}$$

$$H^{2nd} = \frac{2g^2}{N(N-1)} \int_{x,y} \left(f^{aci} A^i_\mu + \frac{1}{2} f^{acd} A^d_\mu \right)_x (\mathcal{M}^{-1})^{ab}_{x,y} \left(f^{bcj} A^j_\mu + \frac{1}{2} f^{bce} A^e_\mu \right)_y, \tag{117}$$

$$H^{3rd} = \frac{2g^2}{N(N-1)} \int_{x,y} \left(f^{aci} A^i_\mu - \frac{1}{2} f^{acd} A^d_\mu \right)_x (\mathcal{M}^{-1})^{ab}_{x,y} \left(f^{bcj} A^j_\mu - \frac{1}{2} f^{bce} A^e_\mu \right)_y, \tag{118}$$

$$H^{4th} = \frac{2g^2}{N(N-1)} \int_{x,y} f^{aci} A^i_{\mu,x} (\mathcal{M}^{-1})^{ab}_{x,y} f^{bcj} A^j_{\mu,y} + \xi\, g^2 \int_{x,y} f^{acd} A^d_{\mu,x} (\mathcal{M}^{-1})^{ab}_{x,y} f^{bce} A^e_{\mu,y}$$

$$+ (N-2)\zeta\, g^2 \int_{x,y} f^{aci} A^c_{\mu,x} (\mathcal{M}^{-1})^{ab}_{x,y} f^{bdi} A^d_{\mu,y}, \tag{119}$$

$$H^{5th} = \frac{2g^2}{N(N-1)} \int_{x,y} \left(f^{aci} A^i_\mu + \frac{\alpha}{2} f^{acd} A^d_\mu \right)_x (\mathcal{M}^{-1})^{ab}_{x,y} \left(f^{bcj} A^j_\mu + \frac{\alpha}{2} f^{bce} A^e_\mu \right)_y$$

$$+ \frac{(N-2)g^2}{2N(N-1)}\beta \int_{x,y} f^{aci} A^c_{\mu,x} (\mathcal{M}^{-1})^{ab}_{x,y} f^{bdi} A^d_{\mu,y}. \tag{120}$$

In all expressions (116)–(120), the nonlocal operator $(\mathcal{M}^{-1})^{ab}_{x,y}$ is the inverse of the Faddeev–Popov operator (15) and the symbol $\int_{x,y}$ stands for $\int \mathrm{d}^4 x\, \mathrm{d}^4 y$. In equation (119), the parameters ξ and ζ are positive and obey the following constraint:

$$\xi + \zeta = \frac{1}{2N(N-1)}. \tag{121}$$

In expression (120), the parameter β is positive and is related to α according to

$$\alpha^2 + (N-2)\beta = N - 2. \tag{122}$$

Let us now see how these five candidates fulfill the three requirements. The first criterion claims that, in the quadratic approximation, the horizon function has to reduce to the Gribov term defined in (108). In the quadratic approximation, the inverse of the Faddeev–Popov can be taken as

$$(\mathcal{M}^{-1})^{ab} \approx \frac{\delta^{ab}}{-\partial^2}. \tag{123}$$

Moreover, using the following relations between the structure constants:

$$\begin{aligned} f^{abi} f^{abc} &= 0, & f^{abi} f^{abj} &= N\delta^{ij}, \\ f^{abc} f^{abd} &= (N-2)\delta^{cd}, & f^{aic} f^{aid} &= \delta^{cd}, \end{aligned} \tag{124}$$

one can easily check that all five candidates (116)–(120) obey the first requirement.

The second requirement states that, when $N = 2$, the horizon function must equal expression (114). It is easy to check that this requirement is also fulfilled by observing that in the case of $SU(2)$ we have

$$f^{abi} \rightarrow f^{ab3} = \varepsilon^{ab}, \qquad f^{abc} \rightarrow 0, \qquad A_\mu^i \rightarrow A_\mu^3 = A_\mu. \tag{125}$$

Besides, there are terms proportional to $(N - 2)$ which automatically vanish for $N = 2$.

The third requirement is about the possibility of casting the nonlocal horizon terms in a local form by introducing a suitable set of auxiliary fields, in a way compatible with power counting renormalizability. Let us analyze each case separately. The first candidate (116) consists of the sum of two different terms which can be localized independently as follows:

$$e^{-\gamma^4 H^{1st}} = \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} (\det \mathcal{M})^{4N(N-1)} \exp\left[-\int d^4x \left(\bar{\varphi}_\mu^{ac} \mathcal{M}^{ab} \varphi_\mu^{bc} + \theta^2 f^{abi} A_\mu^i (\varphi - \bar{\varphi})_\mu^{ab} \right) \right]$$
$$\times \int \mathcal{D}\lambda \mathcal{D}\bar{\lambda} (\det \mathcal{M})^{4N(N-1)} \exp\left[-\int d^4x \left(\bar{\lambda}_\mu^{ac} \mathcal{M}^{ab} \lambda_\mu^{bc} + \frac{\theta^2}{2} f^{abc} A_\mu^c (\lambda - \bar{\lambda})_\mu^{ab} \right) \right], \tag{126}$$

where θ^2 stands for

$$\theta^2 = \sqrt{\frac{2g^2\gamma^4}{N(N-1)}}. \tag{127}$$

In this expression, $\left(\varphi_\mu^{ab}, \bar{\varphi}_\mu^{ab}\right)$ and $\left(\lambda_\mu^{ab}, \bar{\lambda}_\mu^{ab}\right)$ are two pairs of complex commuting auxiliary fields. The determinant, $(\det \mathcal{M})^{4N(N-1)}$, can be exponentiated by employing a pair of complex anticommuting fields $\left(\bar{\omega}_\mu^{ac}, \omega_\mu^{bc}\right)$:

$$(\det \mathcal{M})^{4N(N-1)} = \int \mathcal{D}\omega \mathcal{D}\bar{\omega} \, e^{\int d^4x \, \bar{\omega}_\mu^{ac} \mathcal{M}^{ab} \omega_\mu^{bc}}. \tag{128}$$

Thus, the local version of H^{1st} is

$$H_{\text{Local}}^{1st} = \int d^4x \left[\bar{\varphi}_\mu^{ac} \mathcal{M}^{ab} \varphi_\mu^{bc} - \bar{\omega}_\mu^{ac} \mathcal{M}^{ab} \omega_\mu^{bc} + \theta^2 f^{abi} A_\mu^i (\varphi - \bar{\varphi})_\mu^{ab} \right.$$
$$\left. + \bar{\lambda}_\mu^{ac} \mathcal{M}^{ab} \lambda_\mu^{bc} - \bar{\eta}_\mu^{ac} \mathcal{M}^{ab} \eta_\mu^{bc} + \frac{\theta^2}{2} f^{abc} A_\mu^c (\lambda - \bar{\lambda})_\mu^{ab} \right], \tag{129}$$

where $\left(\eta_\mu^{ab}, \bar{\eta}_\mu^{ab}\right)$ is another pair of complex anticommuting fields similar to $\left(\omega_\mu^{ab}, \bar{\omega}_\mu^{ab}\right)$. As one can easily see, the set of fields $(\varphi, \bar{\varphi}, \omega, \bar{\omega})$ is responsible for the localization of the diagonal sector, while the set $(\lambda, \bar{\lambda}, \eta, \bar{\eta})$ allows us to localize the off-diagonal sector.

In a similar way, we can localize the remaining candidates. To localize expressions (117) and (118), it is necessary to employ only one set of localizing fields $(\varphi, \bar{\varphi}, \omega, \bar{\omega})$:

$$H_{\text{Local}}^{2nd} = \int d^4x \left[\bar{\varphi}_\mu^{ac} \mathcal{M}^{ab} \varphi_\mu^{bc} - \bar{\omega}_\mu^{ac} \mathcal{M}^{ab} \omega_\mu^{bc} + \theta^2 \left(f^{abi} A_\mu^i + \frac{1}{2} f^{abc} A_\mu^c \right) (\varphi - \bar{\varphi})_\mu^{ab} \right], \tag{130}$$

$$H_{\text{Local}}^{3rd} = \int d^4x \left[\bar{\varphi}_\mu^{ac} \mathcal{M}^{ab} \varphi_\mu^{bc} - \bar{\omega}_\mu^{ac} \mathcal{M}^{ab} \omega_\mu^{bc} + \theta^2 \left(f^{abi} A_\mu^i - \frac{1}{2} f^{abc} A_\mu^c \right) (\varphi - \bar{\varphi})_\mu^{ab} \right]. \tag{131}$$

In the case of expression (119), we find

$$H_{\text{Local}}^{4th} = \int d^4x \left[\bar{\varphi}_\mu^{ac} \mathcal{M}^{ab} \varphi_\mu^{bc} - \bar{\omega}_\mu^{ac} \mathcal{M}^{ab} \omega_\mu^{bc} + \theta^2 f^{abi} A_\mu^i (\varphi - \bar{\varphi})_\mu^{ab} \right.$$
$$+ \bar{\lambda}_\mu^{ac} \mathcal{M}^{ab} \lambda_\mu^{bc} - \bar{\eta}_\mu^{ac} \mathcal{M}^{ab} \eta_\mu^{bc} + \xi^{1/2} \gamma^2 \, g f^{abc} A_\mu^c (\lambda - \bar{\lambda})_\mu^{ab}$$
$$\left. + \bar{\psi}_\mu^{ai} \mathcal{M}^{ab} \psi_\mu^{bi} - \bar{\chi}_\mu^{ai} \mathcal{M}^{ab} \chi_\mu^{bi} + \sqrt{(N-2)\xi} \, \gamma^2 \, g f^{abi} A_\mu^a (\psi - \bar{\psi})_\mu^{bi} \right]. \tag{132}$$

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Note that, here, a new set of fields has been employed: $(\psi_\mu^{ai}, \bar\psi_\mu^{ai})$ stands for a pair of complex commuting fields and $(\chi_\mu^{ai}, \bar\chi_\mu^{ai})$ for a pair of complex anticommuting ones. We also observe that these fields carry both off-diagonal and diagonal indices.

Finally, the local version of (120) is

$$
H_{\text{Local}}^{\text{5th}} = \int \mathrm{d}^4 x \left[\bar\varphi_\mu^{ac} \mathcal{M}^{ab} \varphi_\mu^{bc} - \bar\omega_\mu^{ac} \mathcal{M}^{ab} \omega_\mu^{bc} + \theta^2 \left(f^{abi} A_\mu^i + \frac{\alpha}{2} f^{abc} A_\mu^c \right) (\varphi - \bar\varphi)_\mu^{ab} \right.
$$
$$
\left. + \bar\psi_\mu^{ai} \mathcal{M}^{ab} \psi_\mu^{bi} - \bar\chi_\mu^{ai} \mathcal{M}^{ab} \chi_\mu^{bi} + \sqrt{(N-2)\beta} \frac{\theta^2}{2} f^{abi} A_\mu^a (\psi - \bar\psi)_\mu^{bi} \right]. \tag{133}
$$

It is worth to point out that all local expressions, equations (129), (130), (131), (132), (133), are power counting renormalizable. All five candidates, (116)–(120), fulfill the three requirements given at the beginning of this section.

A further requirement is thus needed in order to select only one candidate. The natural setup is that of strengthening the requirement of the equivalence between Gribov's form factor $\sigma(0; A)$ and the possible horizon terms, as done in [51] in the case of the Landau gauge. This amounts to expand the horizon terms, equations (116)–(120), till the third order in the gauge field and compare the resulting expressions with that which we have already obtained for $\sigma(0; A)$; see equation (96).

6.3. Selecting only one candidate for the horizon function

In this section we show that the only candidate which is compatible with the expression of $\sigma(0; A)$ is H^{2nd} (equation (117)). For that purpose, the expansion till the first order in the gauge field of the inverse, $(\mathcal{M}^{-1})^{ab}$, of the Faddeev–Popov operator

$$
\mathcal{M}^{ab}(x)(\mathcal{M}^{-1})^{bc}(x, y) = \delta^{ac} \delta(x - y) \tag{134}
$$

is needed. Recalling that $\mathcal{M}^{ab}(x)$ is defined by expression (15), we can write

$$
\mathcal{M}^{ab} = -\delta^{ab} \partial^2 + 2g f^{abi} A_\mu^i \partial_\mu + g f^{abc} A_\mu^c \partial_\mu + O(g^2). \tag{135}
$$

To evaluate the inverse of $\mathcal{M}^{ab}$ order-by-order, we set

$$
(\mathcal{M}^{-1})^{ab}(x, y) = \mathcal{G}_0^{ab}(x - y) + g \mathcal{G}_1^{ab}(x, y) + O(g^2), \tag{136}
$$

so that, to the first order, we have to solve the equation

$$
\mathcal{M}^{ac}(x)\left(\mathcal{G}_0^{ab}(x - y) + g \mathcal{G}_1^{ab}(x, y) + O(g^2)\right) = \delta^{ab}\delta(x - y), \tag{137}
$$

where $\mathcal{G}_0^{ab}(x - y)$ is the free ghost propagator:

$$
\mathcal{G}_0^{ab}(x - y) = \frac{\delta^{ab}}{|x - y|^2} = \delta^{ab} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \mathrm{e}^{-iq(x-y)}. \tag{138}
$$

To the first order, we have

$$
-\partial_x^2 \mathcal{G}_1^{ab}(x, y) = -2 f^{aci} A_\mu^i(x) \partial_\mu^x \mathcal{G}_0^{cb}(x - y) - f^{acd} A_\mu^d(x) \partial_\mu^x \mathcal{G}_0^{cb}(x - y), \tag{139}
$$

which gives

$$
\mathcal{G}_1^{ab}(x, y) = -2 \int \mathrm{d}^4 z \frac{1}{|x - z|^2} \left(f^{abi} A_\mu^i(z) + \frac{1}{2} f^{abc} A_\mu^c(z) \right) \partial_\mu^z \frac{1}{|z - y|^2}. \tag{140}
$$

Therefore,

$$
(\mathcal{M}^{-1})^{ab}(x, y) = \frac{\delta^{ab}}{|x - y|^2} - 2 \int \mathrm{d}^4 z \frac{1}{|x - z|^2} \left(f^{abi} A_\mu^i(z) + \frac{1}{2} f^{abc} A_\mu^c(z) \right) \partial_\mu^z \frac{1}{|z - y|^2}
$$
$$
+ O(g^2). \tag{141}
$$

We are now ready to expand the horizon function (117) in powers of the gauge field. Introducing the combination

$$B_\mu^{ab} = f^{abi} A_\mu^i + \tfrac{1}{2} f^{abc} A_\mu^c, \tag{142}$$

one can write

$$H^{\text{2nd}} = \frac{2g^2}{N(N-1)} \int \mathrm{d}^4x\, \mathrm{d}^4y\, B_\mu^{ac}(x)(\mathcal{M}^{-1})^{ab}(x,y) B_\mu^{bc}(y), \tag{143}$$

which, by means of equations (141) and (142), becomes

$$H^{\text{2nd}} = \frac{2g^2}{N(N-1)} \int \mathrm{d}^4x\, \mathrm{d}^4y\, B_\mu^{ab}(x) \frac{1}{|x-y|^2} B_\mu^{ab}(y) - \frac{4g^3}{N(N-1)}$$
$$\times \int \mathrm{d}^4x\, \mathrm{d}^4y\, \mathrm{d}^4z\, B_\mu^{ac}(x) \frac{1}{|x-z|^2} B_\nu^{ab}(z) \partial_\nu^z \frac{1}{|z-y|^2} B_\mu^{bc}(y) + O(g^4). \tag{144}$$

Explicitly, we have

$$H^{\text{2nd}} = \frac{2g^2}{N-1} \int \mathrm{d}^4x\, \mathrm{d}^4y\, \frac{A_\mu^i(x) A_\mu^i(y)}{|x-y|^2} + \frac{(N-2)g^2}{2N(N-1)} \int \mathrm{d}^4x\, \mathrm{d}^4y\, \frac{A_\mu^a(x) A_\mu^a(y)}{|x-y|^2}$$
$$- \frac{4g^3}{N(N-1)} \int \mathrm{d}^4x\, \mathrm{d}^4y\, \mathrm{d}^4z\, \frac{1}{|x-z|^2} \partial_\nu^z \frac{1}{|z-y|^2} \left(f^{aci} f^{abj} f^{bck} A_\mu^i(x) A_\nu^j(z) A_\mu^k(y) \right.$$
$$+ \tfrac{1}{2} f^{aci} f^{abj} f^{bcf} A_\mu^i(x) A_\nu^j(z) A_\mu^f(y) + \tfrac{1}{2} f^{aci} f^{abe} f^{bck} A_\mu^i(x) A_\nu^e(z) A_\mu^k(y)$$
$$+ \tfrac{1}{4} f^{aci} f^{abe} f^{bcf} A_\mu^i(x) A_\nu^e(z) A_\mu^f(y) + \tfrac{1}{2} f^{acd} f^{abj} f^{bck} A_\mu^d(x) A_\nu^j(z) A_\mu^k(y)$$
$$+ \tfrac{1}{4} f^{acd} f^{abj} f^{bcf} A_\mu^d(x) A_\nu^j(z) A_\mu^f(y) + \tfrac{1}{4} f^{acd} f^{abe} f^{bck} A_\mu^d(x) A_\nu^e(z) A_\mu^k(y)$$
$$\left. + \tfrac{1}{8} f^{acd} f^{abe} f^{bcf} A_\mu^d(x) A_\nu^e(z) A_\mu^f(y) \right) + O(g^4). \tag{145}$$

Taking the Fourier transform

$$H^{\text{2nd}} = \frac{2g^2}{(N-1)} \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{A_\lambda^i(q) A_\lambda^i(-q)}{q^2} + \frac{(N-2)g^2}{2N(N-1)} \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{A_\lambda^a(q) A_\lambda^a(-q)}{q^2}$$
$$- \frac{2ig^3}{N(N-1)} \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{p_\mu}{p^2(p+q)^2} \left(2 f^{abi} f^{bcj} f^{cak} A_\mu^i(-p) A_\nu^j(-q) A_\mu^k(p+q) \right.$$
$$+ f^{abi} f^{bdj} f^{dac} A_\mu^i(-p) A_\nu^j(-q) A_\mu^c(p+q)$$
$$+ f^{abi} f^{bdc} f^{daj} A_\mu^i(-p) A_\nu^c(-q) A_\mu^j(p+q)$$
$$+ f^{abc} f^{bdi} f^{daj} A_\mu^c(-p) A_\nu^i(-q) A_\mu^j(p+q)$$
$$+ \tfrac{1}{2} f^{bcd} f^{bde} f^{dai} A_\mu^c(-p) A_\nu^e(-q) A_\mu^i(p+q)$$
$$+ \tfrac{1}{2} f^{abc} f^{bdi} f^{dae} A_\mu^c(-p) A_\nu^i(-q) A_\mu^e(p+q)$$
$$+ \tfrac{1}{2} f^{abi} f^{bdc} f^{dae} A_\mu^i(-p) A_\nu^c(-q) A_\mu^e(p+q)$$
$$\left. + \tfrac{1}{4} f^{abc} f^{bde} f^{daf} A_\mu^d(-p) A_\nu^e(-q) A_\mu^f(p+q) \right) + O(g^4), \tag{146}$$

and comparing with Gribov's form factor $\sigma(0; A)$, equation (96), it is apparent that

$$\boxed{\sigma(0, A) = \tfrac{1}{2} H^{\text{2nd}} + O(A^4).} \tag{147}$$

The same procedure can be easily repeated for the other expressions, H^{1st}, H^{3st}, H^{4st}, H^{5st}. The corresponding expansions do not coincide with equation (96). Therefore, from now on, we shall consider expression (143) as the correct horizon function we were looking for. Consequently, for the partition function defining the theory we shall take

$$\mathcal{Z} = \int d\mu \, e^{-(S_{YM} + \gamma^4 H^{2nd})}. \tag{148}$$

In the next section, we shall construct a complete local action out of H^{2nd} and we shall study the possible dimension 2 operators which can be introduced. The effect of the condensation of these operators on the tree-level gluon and ghost propagators will also be worked out.

7. Local dimension 2 operators and the tree-level gluon and ghost propagators

7.1. A local and invariant action

According to expression (148), the general local action that we can construct for the $SU(N)$ Euclidean Yang–Mills theory in the maximal Abelian gauge reads

$$\mathcal{Z} = \int \mathcal{D}\Phi \, e^{-S_0}, \qquad S_0 = S_{YM} + S_{MAG} + S_{Local}, \tag{149}$$

where

$$\mathcal{D}\Phi \equiv \mathcal{D}A^a \mathcal{D}A^i \mathcal{D}b^a \mathcal{D}b^i \mathcal{D}c^a \mathcal{D}\bar{c}^a \mathcal{D}c^i \mathcal{D}\bar{c}^i \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \mathcal{D}\omega \mathcal{D}\bar{\omega}, \tag{150}$$

and $S_{Local} \equiv H_{Local}^{2nd}$ given by equation (130). Explicitly, we have

$$\begin{aligned}
S_0 = \int d^4x \Bigg(&\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{4} F_{\mu\nu}^i F_{\mu\nu}^i + ib^a \, D_\mu^{ab} A_\mu^b - \bar{c}^a \mathcal{M}^{ab} c^b - \left(D_\mu^{ad} A_\mu^d \right) \\
&\times \left(g f^{abc} \bar{c}^b c^c + g f^{abi} \bar{c}^b c^i \right) + ib^i \, \partial_\mu A_\mu^i + \bar{c}^i \, \partial_\mu \left(\partial_\mu c^i + g f^{abi} A_\mu^a c^b \right) + \bar{\varphi}_\mu^{ac} \mathcal{M}^{ab} \varphi_\mu^{bc} \\
&- \bar{\omega}_\mu^{ac} \mathcal{M}^{ab} \omega_\mu^{bc} + \sqrt{\frac{2\gamma^4}{N(N-1)}} \left(g f^{abi} A_\mu^i + \frac{g}{2} f^{abc} A_\mu^c \right) \left(\varphi_\mu^{ab} - \bar{\varphi}_\mu^{ab} \right) \Bigg).
\end{aligned} \tag{151}$$

Having at our disposal a local action, we may investigate its symmetry content. Let us start with the BRST symmetry, already obtained for the gauge-fixed theory; see equations (16). As established in the case of the Landau gauge [7, 8, 13] and $SU(2)$ maximal Abelian gauge [49], the auxiliary fields $(\varphi, \bar{\varphi}, \omega, \bar{\omega})$ transform as a BRST quartet, i.e.

$$\begin{aligned}
s\varphi_\mu^{ab} &= \omega_\mu^{ab}, & s\omega_\mu^{ab} &= 0, \\
s\bar{\omega}_\mu^{ab} &= \bar{\varphi}_\mu^{ab}, & s\bar{\varphi}_\mu^{ab} &= 0, \\
s^2 &= 0.
\end{aligned} \tag{152}$$

As a consequence, we obtain

$$\begin{aligned}
s S_{Local} &= s S^{(1)} + s S^{(2)}, \\
s S^{(1)} &= s \int d^4x \left(\bar{\varphi}_\mu^{ac} \mathcal{M}^{ab} \varphi_\mu^{bc} - \bar{\omega}_\mu^{ac} \mathcal{M}^{ab} \omega_\mu^{bc} \right) = \int d^4x \left(\bar{\varphi}_\mu^{ac} \mathcal{N}^{ab} \varphi_\mu^{bc} + \bar{\omega}_\mu^{ac} \mathcal{N}^{ab} \omega_\mu^{bc} \right), \\
s S^{(2)} &= \theta^2 s \int d^4x \, B_\mu^{ab} \left(\varphi_\mu^{ab} - \bar{\varphi}_\mu^{ab} \right) = \theta^2 \int d^4x \left[\left(s B_\mu^{ab} \right) \left(\varphi_\mu^{ab} - \bar{\varphi}_\mu^{ab} \right) + B_\mu^{ab} \omega_\mu^{ab} \right],
\end{aligned} \tag{153}$$

26

where θ^2 is given in equation (127), $B_\mu^{ab}(x)$ in equation (142) and the operator $\mathcal{N}^{ab}$ is defined as

$$
\begin{aligned}
\mathcal{N}^{ab}\Phi^b &= s(\mathcal{M}^{ab}\Phi^b) - \mathcal{M}^{ab}s\Phi^b \\
&= \big[2gf^{abi}(sA_\mu^i)\partial_\mu + gf^{abc}(sA_\mu^c)\partial_\mu + gf^{abi}\partial_\mu(sA_\mu^i) + 2g^2 f^{aci}f^{bcj}(sA_\mu^i)A_\mu^j \\
&\quad g^2 f^{adc}f^{bdi}s(A_\mu^c A_\mu^i) - g^2(f^{aci}f^{bdi} + f^{adi}f^{bci})(sA_\mu^c)A_\mu^d\big]\Phi^b \\
&= -\big[2gf^{aci}(\partial_\mu c^i + gf^{cdi}A_\mu^c c^d)D_\mu^{cb} - gf^{acd}(D_\mu^{ce}c^e + gf^{cef}A_\mu^e c^f + gf^{cei}A_\mu^e c^i)D_\mu^{db} \\
&\quad + gf^{abi}\partial_\mu(\partial_\mu c^i + gf^{cdi}A_\mu^c c^d) + g^2 f^{adc}f^{bdi}A_\mu^c(\partial_\mu c^i + gf^{efi}A_\mu^e c^f) \\
&\quad - g^2(f^{aci}f^{bdi} + f^{adi}f^{bci})(D_\mu^{ce}c^e + gf^{cef}A_\mu^e c^f + gf^{cej}A_\mu^e c^j)A_\mu^d\big]\Phi^b,
\end{aligned}
\tag{154}
$$

with Φ^a representing all the off-diagonal fields of the theory

$$
\Phi^a \equiv \big\{A_\mu^a, b^a, c^a, \bar{c}^a, \varphi_\mu^{ab}, \bar{\varphi}_\mu^{ab}, \omega_\mu^{ab}, \bar{\omega}_\mu^{ab}\big\}.
\tag{155}
$$

The first term $S^{(1)}$ can be rewritten in a BRST invariant way by making a linear shift in the variable $\omega_\mu^{ab}(x)$ [7, 8, 13, 49]:

$$
\omega_\mu^{ab}(x) \rightarrow \omega_\mu^{ab}(x) + \int d^4 y (\mathcal{M}^{-1})^{ac}(x,y)\mathcal{N}^{cd}(y)\varphi_\mu^{db}(y).
\tag{156}
$$

Thus,

$$
\begin{aligned}
S^{(1)} \rightarrow S_{\text{inv}}^{(1)} &= \int d^4 x \big(\bar{\varphi}_\mu^{ac}\mathcal{M}^{ab}\varphi_\mu^{bc} - \bar{\omega}_\mu^{ac}\mathcal{M}^{ab}\omega_\mu^{bc} - \bar{\omega}_\mu^{ac}\mathcal{N}^{ab}\varphi_\mu^{bc}\big) \\
&= \int d^4 x \big[\bar{\varphi}_\mu^{ac}\mathcal{M}^{ab}\varphi_\mu^{bc} - \bar{\omega}_\mu^{ac}\mathcal{M}^{ab}\omega_\mu^{bc} - \bar{\omega}_\mu^{ac}\big(s(\mathcal{M}^{ab}\varphi_\mu^{bc}) - \mathcal{M}^{ab}\omega_\mu^{bc}\big)\big] \\
&= \int d^4 x \big[\bar{\varphi}_\mu^{ac}\mathcal{M}^{ab}\varphi_\mu^{bc} - \bar{\omega}_\mu^{ac}s(\mathcal{M}^{ab}\varphi_\mu^{bc})\big] \\
&= s\int d^4 x \,\bar{\omega}_\mu^{ac}\mathcal{M}^{ab}\varphi_\mu^{bc}.
\end{aligned}
\tag{157}
$$

In order to deal with the second term $S^{(2)}$, let us note that

$$
B_\mu^{ab}\bar{\varphi}_\mu^{ab} = B_\mu^{ab}s\bar{\omega}_\mu^{ab} = s\big(B_\mu^{ab}\bar{\omega}_\mu^{ab}\big) - \big(sB_\mu^{ab}\big)\bar{\omega}_\mu^{ab},
\tag{158}
$$

and then,

$$
S^{(2)} = \theta^2 \int d^4 x \big[B_\mu^{ab}\varphi_\mu^{ab} - s\big(B_\mu^{ab}\bar{\omega}_\mu^{ab}\big) + \big(sB_\mu^{ab}\big)\bar{\omega}_\mu^{ab}\big].
\tag{159}
$$

The last term on the rhs of (159) can be eliminated by performing a second linear shift in $\omega_\mu^{ab}(x)$ [7, 8, 13, 49]:

$$
\omega_\mu^{ab}(x) \rightarrow \omega_\mu^{ab}(x) + \theta^2 \int d^4 y (\mathcal{M}^{-1})^{ac}(x,y)\, sB_\mu^{cb}(y).
\tag{160}
$$

Therefore, the local version of the horizon function might be taken as

$$
S_{\text{Local}} = s\int d^4 x \,\bar{\omega}_\mu^{ac}\mathcal{M}^{ab}\varphi_\mu^{bc} + \theta^2 \int d^4 x \big[B_\mu^{ab}\varphi_\mu^{ab} - s\big(B_\mu^{ab}\bar{\omega}_\mu^{ab}\big)\big].
\tag{161}
$$

Consequently, the action S_0 becomes

$$
\begin{aligned}
S_0 &= \int d^4 x \Big(\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{4}F_{\mu\nu}^i F_{\mu\nu}^i + ib^a D_\mu^{ab}A_\mu^b - \bar{c}^a \mathcal{M}^{ab}c^b \\
&\quad - \big(D_\mu^{ad}A_\mu^d\big)\big(gf^{abc}\bar{c}^b c^c + gf^{abi}\bar{c}^b c^i\big) + ib^i \partial_\mu A_\mu^i + \bar{c}^i \partial_\mu\big(\partial_\mu c^i + gf^{abi}A_\mu^a c^b\big)\Big) \\
&\quad + s\int d^4 x \,\bar{\omega}_\mu^{ac}\mathcal{M}^{ab}\varphi_\mu^{bc} + \theta^2 \int d^4 x \big[B_\mu^{ab}\varphi_\mu^{ab} - s\big(B_\mu^{ab}\bar{\omega}_\mu^{ab}\big)\big].
\end{aligned}
\tag{162}
$$

As it occurs in the case of the Landau gauge [7, 8, 13] and $SU(2)$ maximal Abelian gauge [49], the action S_0 does not possess exact BRST invariance, which turns out to be broken by soft terms proportional to the Gribov parameter, i.e. to θ^2. In fact

$$s S_0 = \theta^2 \Delta_{\text{break}}, \tag{163}$$

where

$$\Delta_{\text{break}} = \int \mathrm{d}^4 x \bigg[B_\mu^{ab} \omega_\mu^{ab} - \bigg(f^{abi} \big(\partial_\mu c^i + g f^{cdi} A_\mu^c c^d \big) \\ + \frac{1}{2} f^{abc} \big(D_\mu^{cm} c^m + g f^{cmn} A_\mu^m c^n + g f^{cmi} A_\mu^m c^i \big) \bigg) \varphi_\mu^{ab} \bigg]. \tag{164}$$

Being of dimension 2, the breaking term Δ_{break} is soft. As extensively analyzed in [13, 52], the existence of such a breaking does not jeopardize the renormalizability of the theory as well as the introduction of meaningful operators which display good analyticity properties. Of course, the fact that the breaking is soft is of pivotal importance here. As underlined in [13, 53], the existence of such a breaking is linked to the presence of a boundary in field space, namely the Gribov horizon $\partial\Omega$.

7.2. Embedding the theory into a more general one

The standard way of dealing with soft breaking terms is that of introducing them into the starting action as composite operators coupled to a suitable set of external sources. In doing so, a more general action displaying exact BRST invariance is obtained. Furthermore, the starting action S_0 and its breaking term Δ_{break} are recovered by demanding that the external sources attain a particular value, usually referred to as the physical value [7, 8, 13]. In order to construct such a generalized action, we introduce external sources $\big(V_{\mu\nu}^{ab}, U_{\mu\nu}^{ab}, \bar{V}_{\mu\nu}^{ab}, \bar{U}_{\mu\nu}^{ab} \big)$ transforming as

$$\begin{aligned} s V_{\mu\nu}^{ab} &= U_{\mu\nu}^{ab}, & s U_{\mu\nu}^{ab} &= 0, \\ s \bar{U}_{\mu\nu}^{ab} &= \bar{V}_{\mu\nu}^{ab}, & s \bar{V}_{\mu\nu}^{ab} &= 0, \end{aligned} \tag{165}$$

and the invariant action $S_{\text{Local}}^{\text{inv}}$ given by

$$\begin{aligned} S_{\text{Local}}^{\text{inv}} = {}& s \int \mathrm{d}^4 x \, \bar{\omega}_\mu^{ac} \mathcal{M}^{ab} \varphi_\mu^{bc} + s \int \mathrm{d}^4 x \bigg[\bar{U}_{\mu\nu}^{ac} \bigg(D_\mu^{ab} - \frac{g}{2} f^{abd} A_\mu^d \bigg) \varphi_\nu^{bc} \\ & + V_{\mu\nu}^{ac} \bigg(D_\mu^{ab} - \frac{g}{2} f^{abd} A_\mu^d \bigg) \bar{\omega}_\nu^{bc} \bigg] \\ = {}& \int \mathrm{d}^4 x \bigg\{ \bar{\varphi}_\mu^{ac} \mathcal{M}^{ab} \varphi_\mu^{bc} - \bar{\omega}_\mu^{ac} \mathcal{M}^{ab} \omega_\mu^{bc} - \bar{\omega}_\mu^{ac} \mathcal{N}^{ab} \varphi_\mu^{bc} + V_{\mu\nu}^{ac} \bigg[\bigg(D_\mu^{ab} - \frac{g}{2} f^{abd} A_\mu^d \bigg) \bar{\varphi}_\nu^{bc} \\ & + g f^{abi} \big(\partial_\mu c^i + g f^{dei} A_\mu^d c^e \big) \bar{\omega}_\nu^{bc} + \frac{g}{2} f^{abd} \big(D_\mu^{de} c^e + g f^{def} A_\mu^e c^f + g f^{dei} A_\mu^e c^i \big) \bar{\omega}_\nu^{bc} \bigg] \\ & + \bar{V}_{\mu\nu}^{ac} \bigg(D_\mu^{ab} - \frac{g}{2} f^{abd} A_\mu^d \bigg) \varphi_\nu^{bc} - \bar{U}_{\mu\nu}^{ac} \bigg[\bigg(D_\mu^{ab} - \frac{g}{2} f^{abd} A_\mu^d \bigg) \omega_\nu^{bc} \\ & + g f^{abi} \big(\partial_\mu c^i + g f^{dei} A_\mu^d c^e \big) \varphi_\nu^{bc} + \frac{g}{2} f^{abd} \big(D_\mu^{de} c^e + g f^{def} A_\mu^e c^f + g f^{dei} A_\mu^e c^i \big) \varphi_\nu^{bc} \bigg] \\ & + U_{\mu\nu}^{ac} \bigg(D_\mu^{ab} - \frac{g}{2} f^{abd} A_\mu^d \bigg) \bar{\omega}_\nu^{bc} \bigg\}. \end{aligned} \tag{166}$$

The expression S_{Local} is thus recovered from the generalized one $S_{\text{Local}}^{\text{inv}}$ when the sources attain their physical values:

$$V_{\mu\nu}^{ab}\big|_{\text{phys}} = -\bar{V}_{\mu\nu}^{ab}\big|_{\text{phys}} = -\sqrt{\frac{2\gamma^4}{N(N-1)}}\,\delta^{ab}\delta_{\mu\nu}, \qquad U_{\mu\nu}^{ab}\big|_{\text{phys}} = \bar{U}_{\mu\nu}^{ab}\big|_{\text{phys}} = 0, \tag{167}$$

so that

$$S_{\text{Local}}^{\text{inv}}\big|_{\text{phys}} = S_{\text{Local}}. \tag{168}$$

We end up thus with a generalized invariant action[11]

$$S_{\text{inv}} = S_{\text{YM}} + S_{\text{MAG}} + S_{\text{Local}}^{\text{inv}}, \tag{169}$$

$$sS_{\text{inv}} = 0, \tag{170}$$

which reduces to S_0 in the physical limit

$$S_{\text{inv}}|_{\text{phys}} = S_0. \tag{171}$$

The BRST invariance enjoyed by S_{inv} can be translated in a powerful functional identity, the Slavnov–Taylor identity, which is the starting point for the analysis of the all-order renormalizability, which we shall present separately in a forthcoming work. To derive such identity, we need to introduce more external sources coupled to the nonlinear BRST transformations of the fields [54]. To that purpose, we note that the BRST transformation of the off-diagonal component of the gauge field $A_\mu^a(x)$ can be split into three nonlinear parts:

$$sA_\mu^a = \mathcal{P}_\mu^a + \mathcal{Q}_\mu^a + \mathcal{R}_\mu^a, \tag{172}$$

$$\mathcal{P}_\mu^a = -D_\mu^{ab}c^b, \qquad \mathcal{Q}_\mu^a = -gf^{abc}A_\mu^b c^c, \qquad \mathcal{R}_\mu^a = -gf^{abi}A_\mu^b c^i. \tag{173}$$

These nonlinear terms can be defined separately by introducing the following set of external sources

$$\begin{aligned} s\xi_\mu^a &= K_\mu^a - \Omega_\mu^a, & sK_\mu^a &= s\Omega_\mu^a = 0, \\ s\vartheta_\mu^a &= \Upsilon_\mu^a - \Omega_\mu^a, & s\Upsilon_\mu^a &= s\Omega_\mu^a = 0, \end{aligned} \tag{174}$$

and writing

$$S_{\text{ext}}^{(1)} = \int d^4x \left[\Omega_\mu^a \mathcal{P}_\mu^a + K_\mu^a \mathcal{Q}_\mu^a + \Upsilon_\mu^a \mathcal{R}_\mu^a + \xi_\mu^a (s\mathcal{Q}_\mu^a) + \vartheta_\mu^a (s\mathcal{R}_\mu^a) \right]. \tag{175}$$

Since

$$s^2 A_\mu^a = s\mathcal{P}^a + s\mathcal{Q}^a + s\mathcal{R}^a = 0, \tag{176}$$

one immediately verifies that $sS_{\text{ext}}^{(1)} = 0$. The remaining nonlinear transformations can be defined by

$$S_{\text{ext}}^{(2)} = \int d^4x \left[\Omega_\mu^i (sA_\mu^i) + L^a(sc^a) + L^i(sc^i) \right], \tag{177}$$

with

$$s\Omega_\mu^i = sL^i = sL^a = 0. \tag{178}$$

---

[11] The complete invariant action must contain a generalized version of the term S_α in (17) in order to establish its renormalizability. We refer the reader to [48], where this analysis is done in detail for the $SU(2)$ case. However, in the present section, we are ultimately interested only in the computation of the tree-level propagators of the maximal Abelian gauge; therefore, we can set $\alpha = 0$ at this level.

Therefore, the complete action Σ_0 defined by

$$\Sigma_0 = S_{\mathrm{inv}} + S_{\mathrm{ext}}^{(1)} + S_{\mathrm{ext}}^{(2)} \tag{179}$$

obeys the following Slavnov–Taylor identity:

$$\mathcal{S}(\Sigma_0) = \int \mathrm{d}^4 x \left[\left(\frac{\delta\Sigma_0}{\delta\Omega_\mu^a} + \frac{\delta\Sigma_0}{\delta K_\mu^a} + \frac{\delta\Sigma_0}{\delta\Upsilon_\mu^a} \right) \frac{\delta\Sigma_0}{\delta A_\mu^a} + \frac{\delta\Sigma_0}{\delta\Omega_\mu^i} \frac{\delta\Sigma_0}{\delta A_\mu^i} + \frac{\delta\Sigma_0}{\delta L^a} \frac{\delta\Sigma_0}{\delta c^a} + \frac{\delta\Sigma_0}{\delta L^i} \frac{\delta\Sigma_0}{\delta c^i} \right.$$
$$+ \mathrm{i}b^a \frac{\delta\Sigma_0}{\delta\bar{c}^a} + \mathrm{i}b^i \frac{\delta\Sigma_0}{\delta\bar{c}^i} + \omega_\mu^{ab} \frac{\delta\Sigma_0}{\delta\varphi_\mu^{ab}} + \bar{\varphi}_\mu^{ab} \frac{\delta\Sigma_0}{\delta\bar{\omega}_\mu^{ab}} + U_{\mu\nu}^{ab} \frac{\delta\Sigma_0}{\delta V_{\mu\nu}^{ab}} + \bar{V}_{\mu\nu}^{ab} \frac{\delta\Sigma_0}{\delta\bar{U}_{\mu\nu}^{ab}}$$
$$\left. + \left(K_\mu^a - \Omega_\mu^a \right) \frac{\delta\Sigma_0}{\delta\xi_\mu^a} + \left(\Upsilon_\mu^a - \Omega_\mu^a \right) \frac{\delta\Sigma_0}{\delta\vartheta_\mu^a} \right] = 0. \tag{180}$$

Besides the Slavnov–Taylor identity, equation (180), the action Σ_0 displays a rather rich symmetry content. There exist in fact several symmetries involving the exchange between the Faddeev–Popov ghosts (c^a, $\bar{c}^a$) and the localizing auxiliary fields (φ, $\bar{\varphi}$, ω, $\bar{\omega}$), namely

$$\delta_\mu^a \Sigma_0 = 0, \qquad \bar{\delta}_\mu^a \Sigma_0 = 0, \qquad d_\mu^a \Sigma_0 = 0, \qquad \bar{d}_\mu^a \Sigma_0 = 0, \tag{181}$$

where the operators $\left(\delta_\mu^a, \bar{\delta}_\mu^a, \mathrm{d}_\mu^a, \bar{d}_\mu^a \right)$ act on the fields and sources as

- the δ_μ^a-transformation:

$$\begin{aligned} \delta_\nu^c \bar{c}^a &= \varphi_\nu^{ac}, & \delta_\nu^c \bar{\varphi}_\mu^{ab} &= \delta_{\mu\nu}\delta^{bc} c^a, & \delta_\nu^c b^a &= -\mathrm{i}g f^{abi} \varphi_\nu^{bc} c^i, \\ \delta_\nu^c \Omega_\mu^a &= V_{\mu\nu}^{ac}, & \delta_\nu^c K_\mu^a &= \tfrac{1}{2} V_{\mu\nu}^{ac}, \end{aligned} \tag{182}$$

- the $\bar{\delta}_\mu^a$-transformation:

$$\begin{aligned} \bar{\delta}_\nu^c \bar{c}^a &= \bar{\omega}_\nu^{ac}, \\ \bar{\delta}_\nu^c \omega_\mu^{ab} &= -\delta_{\mu\nu}\delta^{bc} c^a, \\ \bar{\delta}_\nu^c b^a &= -\mathrm{i}g f^{abd} \bar{\omega}_\nu^{bc} c^d - \mathrm{i}g f^{abi} \bar{\omega}_\nu^{bc} c^i, \\ \bar{\delta}_\nu^c \Omega_\mu^a &= \bar{U}_{\mu\nu}^{ac}, \\ \bar{\delta}_\nu^c K_\mu^a &= \tfrac{1}{2} \bar{U}_{\mu\nu}^{ac}, \end{aligned} \tag{183}$$

- the d_μ^a-transformation:

$$\begin{aligned} d_\nu^c \bar{c}^a &= \omega_\nu^{ac} + g f^{abi} \varphi_\nu^{bc} c^i, \\ d_\nu^c \bar{\omega}_\mu^{ab} &= \delta_{\mu\nu}\delta^{bc} c^a, \\ d_\nu^c \bar{\varphi}_\mu^{ab} &= -\delta_{\mu\nu}\delta^{bc} \left(g f^{adi} c^d c^i + \frac{g}{2} f^{ade} c^d c^e \right), \\ d_\nu^c b^a &= -\mathrm{i}g f^{abi} \omega_\nu^{bc} c^i - \mathrm{i}\frac{g^2}{2} f^{abi} f^{dei} \omega_\nu^{bc} c^d c^e, \\ d_\nu^c \Omega_\mu^a &= U_{\mu\nu}^{ac}, \\ d_\nu^c K_\mu^a &= \frac{1}{2} U_{\mu\nu}^{ac}, \\ d_\nu^c \xi_\mu^a &= -\frac{1}{2} V_{\mu\nu}^{ac}, \\ d_\nu^c \vartheta_\mu^a &= -V_{\mu\nu}^{ac}, \end{aligned} \tag{184}$$

- the $\bar{d}^a_\mu$-transformation:

$$\bar{d}^c_\nu \bar{c}^a = -\bar{\varphi}^{ac}_\nu + g f^{abd} \bar{\omega}^{bc}_\nu c^d + g f^{abi} \bar{\omega}^{bc}_\nu c^i,$$

$$\bar{d}^c_\nu \varphi^{ab}_\mu = -\delta_{\mu\nu} \delta^{bc} c^a,$$

$$\bar{d}^c_\nu \omega^{ab}_\mu = \delta_{\mu\nu} \delta^{bc} \left(g f^{adi} c^d c^i + \frac{g}{2} f^{ade} c^d c^e \right),$$

$$\bar{d}^c_\nu b^a = \mathrm{i} g f^{abd} \bar{\varphi}^{bc}_\nu c^c + \mathrm{i} g f^{abi} \bar{\varphi}^{bc}_\nu c^i - \mathrm{i} g^2 f^{abd} f^{dei} \bar{\omega}^{bc}_\nu c^e c^i$$

$$\qquad - \mathrm{i} \frac{g^2}{2} f^{abi} f^{dei} \bar{\omega}^{bc}_\nu c^d c^e - \mathrm{i} \frac{g^2}{2} f^{abd} f^{def} \bar{\omega}^{bc}_\nu c^e c^f, \qquad (185)$$

$$\bar{d}^c_\nu \Omega^a_\mu = -\bar{V}^{ac}_{\mu\nu},$$

$$\bar{d}^c_\nu K^a_\mu = -\frac{1}{2} \bar{V}^{ac}_{\mu\nu},$$

$$\bar{d}^c_\nu \xi^a_\mu = -\frac{1}{2} \bar{U}^{ac}_{\mu\nu},$$

$$\bar{d}^c_\nu \vartheta^a_\mu = -\bar{U}^{ac}_{\mu\nu}.$$

We can also show that

$$\{\delta^a_\mu, s\} = d^a_\mu, \qquad [\bar{\delta}^a_\mu, s] = \bar{d}^a_\mu, \qquad [d^a_\mu, s] = 0, \qquad \{\bar{d}^a_\mu, s\} = 0. \qquad (186)$$

The symmetries of Σ_0 will help us to determine a suitable set of dimension 2 operators which can be consistently introduced in the theory. This will be the subject of the following section.

7.3. Dimension 2 condensates

In this section we shall briefly discuss about the subject of the dimension 2 condensates, which are the result of the condensation of local dimension 2 operators. One should note that the introduction of the horizon function in its localized form, expression (130), entails the introduction of a dimension 2 condensate. In fact, the gap equation (106) implies that the dimension 2 operator $\left(B^{ab}_\mu \left(\varphi^{ab}_\mu - \bar{\varphi}^{ab}_\mu \right) \right)$ acquires a nonvanishing expectation value, i.e. $\left\langle B^{ab}_\mu \left(\varphi^{ab}_\mu - \bar{\varphi}^{ab}_\mu \right) \right\rangle \neq 0$. An analogous condensate is found in the Landau gauge [7, 8, 10, 12, 13], where the gap equation for the Gribov parameter γ implies that $\left\langle f^{ABC} \left(\phi^{AB}_\mu - \bar{\phi}^{AB}_\mu \right) A^C_\mu \right\rangle \neq 0$, where $\left(\phi^{AB}_\mu, \bar{\phi}^{AB}_\mu \right)$ are the auxiliary fields needed for the localization of the horizon function in the Landau gauge.

Furthermore, in complete analogy with the case of the Landau gauge [10, 12, 13], other dimension 2 condensates have to be taken into account in the maximal Abelian gauge, see also [49] for the particular case of $SU(2)$. More precisely, the following dimension 2 operators can be introduced in a way which preserves renormalizability of the theory as well as its symmetry content:

$$\mathcal{O}_{A^2} = A^a_\mu A^a_\mu, \qquad (187)$$

$$\mathcal{O}^i_{\bar{c} \times c} = g f^{abi} c^a c^b, \qquad (188)$$

$$\mathcal{O}_{\bar{f} f} = \bar{\varphi}^{ab}_\mu \varphi^{ab}_\mu - \bar{\omega}^{ab}_\mu \omega^{ab}_\mu - \bar{c}^a c^a. \qquad (189)$$

The operator (187) is related to the dynamical mass generation for the off-diagonal gluons, a feature which supports the Abelian dominance hypothesis. Its condensation has been established in [44], where a dynamical off-diagonal gluon mass has been reported. The ghost

operator (188) is needed in order to account for the dynamical breaking of the $SL(2, \mathbb{R})$ symmetry present in the ghost sector of the maximal Abelian gauge:

$$\delta\Sigma_0 = 0, \qquad \delta\bar{c}^a = c^a, \qquad \delta b^a = g f^{abi} c^b c^i + \frac{g}{2} f^{abc} c^b c^c. \tag{190}$$

Its condensation has been analyzed recently in [55]. Concerning now the third operator, equation (189), we note that it depends on the auxiliary fields $(\varphi, \bar{\varphi}, \omega, \bar{\omega})$. It is in fact needed to account for the nontrivial dynamics developed by those fields. Besides, one can see that this operator is invariant under the transformations (182)–(185). An analogous operator has been found in the Landau gauge [10, 12, 13], where it has allowed us to reconcile the Gribov–Zwanziger framework with the most recent lattice data on the gluon and ghost propagators [20, 21].

Let us briefly show how these operators can be introduced in the theory, by taking the example of the operator $\mathcal{O}_{A^2}$. The other operators can be handled by following an analogous path. Let us introduce a BRST doublet of external sources:

$$s\lambda = J, \qquad s J = 0, \tag{191}$$

and let us define the following BRST invariant term:

$$\begin{aligned}
S_{A^2} &= s \int \mathrm{d}^4 x \left(\frac{1}{2} \lambda \, \mathcal{O}_{A^2} - \frac{\zeta}{2} \lambda J \right) \\
&= \int \mathrm{d}^4 x \left(\frac{1}{2} J \, \mathcal{O}_{A^2} + \frac{1}{2} \lambda \, s\mathcal{O}_{A^2} - \frac{\zeta}{2} J^2 \right),
\end{aligned} \tag{192}$$

where ζ is a dimensionless constant parameter necessary to account for the ultraviolet divergences affecting the correlation function

$$\langle \mathcal{O}_{A^2}(x) \mathcal{O}_{A^2}(y) \rangle. \tag{193}$$

As discussed in [44], the parameter ζ is uniquely determined by the renormalization group equations. Expression (192) is thus added to the action Σ_0, giving

$$\Sigma_1 = \Sigma_0 + S_{A^2}. \tag{194}$$

Keeping the source $J(x)$ and setting all other external sources to their respective physical values[10], we can introduce the generating functional $\mathcal{W}[J]$ according to

$$\mathrm{e}^{-\mathcal{W}[J]} = \int \mathcal{D}\Phi \, \mathrm{e}^{-\Sigma_0 - \int \mathrm{d}^4 x \left(\frac{1}{2} J \mathcal{O}_{A^2} - \frac{\zeta}{2} J^2 \right)}. \tag{195}$$

The vacuum expectation value of the operator $\mathcal{O}_{A^2}$ is then obtained by differentiating with respect to J:

$$\left. \frac{\delta\mathcal{W}[J]}{\delta J} \right|_{J=0} = -\frac{1}{2} \langle \mathcal{O}_{A^2} \rangle. \tag{196}$$

In practice, solving (196) is a difficult task. A shortcut is usually employed, amounting to make use of the Hubbard–Stratonovich field $\eta(x)$. To introduce this field in the theory, one inserts the unity

$$1 = \mathcal{N} \int \mathcal{D}\eta \, \mathrm{e}^{-\frac{1}{2\zeta} \int \mathrm{d}^4 x \left(\frac{\eta}{g} + \frac{1}{2} \mathcal{O}_{A^2} - \zeta J \right)^2}, \tag{197}$$

---

[12] We note that the external sources $(\Omega^{a,i}_\mu, K^a_\mu, \Upsilon^a_\mu, \xi^a_\mu, \vartheta^a_\mu, L^{a,i})$ and $\lambda(x)$ carry a nonvanishing ghost number, so that their physical values vanish.

where $\mathcal{N}$ is a normalization factor, so that expression (195) becomes

$$e^{-\mathcal{W}[J]} = \int \mathcal{D}\Phi \mathcal{D}\eta \, e^{-\Sigma_\eta + \int d^4x \frac{\eta}{g} J},$$

$$\Sigma_\eta = \Sigma_0 + \int d^4x \left(\frac{\eta^2}{2g^2\zeta} + \frac{\eta}{2g\zeta} \mathcal{O}_{A^2} + \frac{1}{8\zeta}(\mathcal{O}_{A^2})^2 \right). \tag{198}$$

With $\mathcal{W}[J]$ written in this way, we can achieve the following relation:

$$\langle \mathcal{O}_{A^2} \rangle = -\frac{2}{g} \langle \eta \rangle, \tag{199}$$

which easily follows from expression (198) upon differentiation with respect to the source J. The advantage of having introduced the Hubbard–Stratonovich field η relies on the fact that the quadratic term J^2 in equation (195) has been replaced by the term Σ_η, equation (198). Also from equation (199) one observes that a nonvanishing vacuum expectation value of the Hubbard–Stratonovich field η gives a nonvanishing condensate $\langle \mathcal{O}_{A^2} \rangle$. It remains thus to find out whether the field η acquires a nonvanishing vacuum expectation value, a task which can be accomplished by evaluating the effective potential corresponding to the action Σ_η. A detailed account of the analysis of the effective potential can be found in [44], where a nonvanishing vacuum expectation value for η has emerged. Setting

$$\eta(x) = \langle \eta \rangle + \tilde{\eta}(x), \qquad \langle \tilde{\eta} \rangle = 0, \tag{200}$$

we obtain

$$\Sigma_\eta = \Sigma_0 + \int d^4x \left(\frac{\langle \eta \rangle^2}{2g^2\zeta} + \frac{\tilde{\eta}^2}{2g^2\zeta} + \frac{\langle \eta \rangle}{2g\zeta} \mathcal{O}_{A^2} + \frac{\tilde{\eta}}{2g\zeta} \mathcal{O}_{A^2} + \frac{1}{8\zeta}(\mathcal{O}_{A^2})^2 \right). \tag{201}$$

Introducing thus the gluon mass

$$m^2 = \frac{\langle \eta \rangle}{g\zeta}, \tag{202}$$

we can also write

$$\Sigma_\eta = \Sigma_0 + \int d^4x \left(\frac{\zeta m^4}{2} + \frac{\tilde{\eta}^2}{2g^2\zeta} + \frac{m^2}{2} \mathcal{O}_{A^2} + \frac{\tilde{\eta}}{2g\zeta} \mathcal{O}_{A^2} + \frac{1}{8\zeta}(\mathcal{O}_{A^2})^2 \right), \tag{203}$$

from which one sees that the condensation of the operator $\mathcal{O}_{A^2}$ results in the dynamical generation of a gluon mass, i.e. $\frac{m^2}{2} A_\mu^a A_\mu^a$. This term will affect the tree-level off-diagonal gluon propagator. In much the same way, the other operators $\mathcal{O}_{\bar{c} \times c}^i$ and $\mathcal{O}_{\bar{f} f}$ will affect the propagators of theory even at the tree level, as we shall see in the next section.

7.4. Tree-level gluon and ghost propagators

In this section we will establish the qualitative behavior of the gluon and ghost propagators by taking into account the effects of the restriction to the Gribov region and of the condensation of the dimension 2 operators (187)–(189), encoded in the following dynamical parameters:

$$\langle \mathcal{O}_{A^2} \rangle \sim m^2, \qquad \langle \mathcal{O}_{\bar{c} \times c}^i \rangle \sim v^i, \qquad \langle \mathcal{O}_{\bar{f} f} \rangle \sim \mu^2. \tag{204}$$

Such parameters will appear in the resulting action as

$$\Sigma = \Sigma_0 + \int d^4x \left(\frac{m^2}{2} \mathcal{O}_{A^2} + v^i \mathcal{O}_{\bar{c} \times c}^i + \mu^2 \mathcal{O}_{\bar{f} f} + \text{'interaction terms'} \right), \tag{205}$$

where Σ is the complete action containing all condensates. In order to evaluate the propagators, it is sufficient to consider only the quadratic terms of Σ:

$$
\Sigma_{\text{quad}} = \lim_{\substack{\alpha \to 0 \\ \beta \to 0}} \int d^4 x \left[\frac{1}{2} A_\mu^a \left((-\partial^2 + m^2)\delta_{\mu\nu} - \frac{1-\alpha}{\alpha} \, \partial_\mu \partial_\nu \right) A_\nu^a + \frac{1}{2} A_\mu^i \left(-\partial^2 \delta_{\mu\nu} - \frac{1-\beta}{\beta} \, \partial_\mu \partial_\nu \right) A_\nu^i \right.
$$
$$
- \bar{c}^a ((-\partial^2 + \mu^2)\delta^{ab} - v^i g f^{abi}) c^b - \bar{c}^i (-\partial^2) c^i + \bar{\varphi}_\mu^{ab}(-\partial^2 + \mu^2)\varphi_\mu^{ab}
$$
$$
\left. - \bar{\omega}_\mu^{ab}(-\partial^2 + \mu^2)\omega_\mu^{ab} + \sqrt{\frac{2g^2\gamma^4}{N(N-1)}} \left(f^{abi} A_\mu^i + \frac{1}{2} f^{abc} A_\mu^c \right) \left(\varphi_\mu^{ab} - \bar{\varphi}_\mu^{ab} \right) \right], \tag{206}
$$

where we have already integrated out the Lagrange multipliers (b^a, b^i) and where we have taken the physical values of the sources $(V, \bar{V}, U, \bar{U})$. A further integration over the auxiliary localizing fields $(\varphi, \bar{\varphi}, \omega, \bar{\omega})$ gives the following expression in momentum space:

$$
\Sigma_{\text{quad}} = \lim_{\substack{\alpha \to 0 \\ \beta \to 0}} \int \frac{d^4 k}{(2\pi)^4} \left(\frac{1}{2} A_\mu^a(k) \, \mathcal{P}_{\mu\nu}^{ab}(k; \alpha) \, A_\nu^b(-k) + \frac{1}{2} A_\mu^i(k) \, \mathcal{Q}_{\mu\nu}^{ij}(k; \beta) \, A_\nu^j(-k) \right.
$$
$$
\left. - c^a(k) \, \mathcal{R}^{ab}(k) \, c^b(-k) - c^i(k) \, \delta^{ij} k^2 \, c^j(-k) \right), \tag{207}
$$

where

$$
\mathcal{P}_{\mu\nu}^{ab}(k; \alpha) = \delta^{ab} \left(\delta_{\mu\nu} \frac{(k^2 + m^2)(k^2 + \mu^2) + (N-2)g^2\gamma^4/N(N-1)}{k^2 + \mu^2} - \frac{1-\alpha}{\alpha} k_\mu k_\nu \right),
$$
$$
\mathcal{Q}_{\mu\nu}^{ij}(k; \beta) = \delta^{ij} \left(\delta_{\mu\nu} \frac{k^2(k^2 + \mu^2) + 4g^2\gamma^4/(N-1)}{k^2 + \mu^2} - \frac{1-\beta}{\beta} k_\mu k_\nu \right), \tag{208}
$$
$$
\mathcal{R}_{\mu\nu}^{ab}(k) = \delta^{ab}(k^2 + \mu^2) - g f^{abi} v^i.
$$

The tree-level propagators of the theory are thus given by

- *the off-diagonal gluon propagator:*

$$
\left\langle A_\mu^a(k) A_\nu^b(-k) \right\rangle = \frac{k^2 + \mu^2}{(k^2 + m^2)(k^2 + \mu^2) + \frac{(N-2)g^2\gamma^4}{N(N-1)}} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \delta^{ab} ; \tag{209}
$$

- *the diagonal gluon propagator:*

$$
\left\langle A_\mu^i(k) A_\nu^j(-k) \right\rangle = \frac{k^2 + \mu^2}{k^2(k^2 + \mu^2) + \frac{4g^2\gamma^4}{(N-1)}} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \delta^{ij} ; \tag{210}
$$

- *the symmetric off-diagonal ghost propagator:*

$$
\langle \bar{c}^a(k) c^b(-k) \rangle_{\text{symm}} = \frac{k^2 + \mu^2}{(k^2 + \mu^2)^2 + \frac{g^2 v^2}{(N-1)}} \, \delta^{ab} ; \tag{211}
$$

- *the antisymmetric off-diagonal ghost propagator:*

$$
\langle \bar{c}^a(k) c^b(-k) \rangle_{\text{antisymm}} = \frac{g f^{abi} v^i}{(k^2 + \mu^2)^2 + \frac{g^2 v^2}{(N-1)}} ; \tag{212}
$$

- *the diagonal ghost propagator:*

$$
\langle \bar{c}^i(k) c^j(-k) \rangle = \frac{1}{k^2} \, \delta^{ij}. \tag{213}
$$

With the exception of the diagonal ghost propagator, equation (213), which exhibits a free behavior, we observe that all remaining propagators turn out to be suppressed in the infrared region, a result which can be seen as a direct generalization of what has been found in the case of $SU(2)$ [49].

The off-diagonal gluon propagator deserves a little comment. Note that it can be written as

$$\left\langle A_\mu^a(k) A_\nu^b(-k) \right\rangle = \left(\frac{1}{k^2 + m^2} - \rho_N(k) \right) \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \delta^{ab}, \tag{214}$$

where $\rho_N(k)$ represents the deviation from the Yukawa-type behavior, and is given by

$$\rho_N(k) = \frac{\frac{(N-2)g^2\gamma^4}{N(N-1)}}{(k^2 + m^2)\left[(k^2 + m^2)(k^2 + \mu^2) + \frac{(N-2)g^2\gamma^4}{N(N-1)} \right]}. \tag{215}$$

The factor $\rho_N(k)$ vanishes for $N = 2$ so that the off-diagonal gluon propagator behaves exactly like the Yukawa propagator, as obtained in [32, 49] for the particular $SU(2)$ case. However, for $N > 2$, it seems to deviate from a pure Yukawa behavior, a feature which would be worth to investigate by lattice simulations in the relevant case of $SU(3)$.

8. Conclusions

In this work we have studied a few aspects of the issue of the Gribov copies in $SU(N)$ Euclidean Yang–Mills theories in the maximal Abelian gauge. The so-called Gribov region Ω has been introduced and some of its properties have been established. Summarizing, the region Ω is convex, bounded in all off-diagonal directions and unbounded in all diagonal ones.

The implementation of the restriction of the domain of integration in the functional integral to the region Ω has been considered. A careful study of the horizon function has been provided. In particular, we have shown that use of Gribov's no pole condition allows us to select only one candidate, given by expression (117). A local action from the restriction to the region Ω, equation (130), has been constructed and its symmetry content established.

It is worth mentioning that the results obtained in [32, 47, 49] for the particular case of $SU(2)$ can be completely recovered by setting $N = 2$, which provides a very good check.

A detailed analysis of the propagators of the theory has been performed. The general case of $SU(N)$, $N > 2$, displays differences with respect to the case of $SU(2)$. In particular, as can be observed from expression (209), the off-diagonal component of the gluon propagator turns out to be affected by the restriction to Ω as well as by the condensation of the operators (187)–(189). The diagonal gluon propagator, equation (210), exhibits a Gribov–Stingl-type behavior depending on the Gribov parameter γ and on the parameter μ corresponding to the condensation of the operator (189). The symmetric off-diagonal ghost propagator, equation (211), turns out to be dependent from the parameters μ and $v^2 = v^i v^i$ while, as expected, the antisymmetric off-diagonal ghost propagator, equation (212), turns out to be directly proportional to the parameter v^i stemming from the ghost condensate $\left\langle \mathcal{O}_{\text{ghost}}^i \right\rangle \sim v^i$.

All propagators are seen to be suppressed in the infrared region. Moreover, they are nonvanishing at $k = 0$. Although numerical studies of the gluon and ghost propagators in the maximal Abelian gauge have been performed only in the case of $SU(2)$, these features seem to be in very good agreement with the most recent numerical data [41]. From this point of view, it would rather be interesting to perform a numerical study of $SU(3)$ in order to check our prevision.

Acknowledgments

Appendix. Details on the calculation of the third-order off-diagonal ghost correlation function

In order to evaluate the off-diagonal ghost correlation function $\langle \bar{c}^a(x) c^b(y) \rangle$, let us start with the usual Gell-Mann–Low formula theory:

$$\langle \bar{c}^a(x) c^b(y) \rangle = \langle 0 | \bar{c}_0^a(x) c_0^b(y) \, \mathrm{e}^{-S_{\mathrm{int}}[\bar{c}_0, c_0]} | 0 \rangle, \tag{A.1}$$

where the fields appearing on the right-hand side of equation (A.1) are free fields and $S_{\mathrm{int}}[\bar{c}, c]$ stands for the interaction term given by

$$S_{\mathrm{int}}[\bar{c}, c] = - \int \mathrm{d}^4 x \big(2g f^{abi} A_\mu^i \bar{c}^a \partial_\mu c^b + g f^{abc} A_\mu^c \bar{c}^a \partial_\mu c^b + g^2 f^{aci} f^{bcj} A_\mu^i A_\mu^j \bar{c}^a c^b$$
$$+ g^2 f^{adc} f^{bdi} A_\mu^c A_\mu^i \bar{c}^a c^b - g^2 f^{cai} f^{dbi} A_\mu^c A_\mu^d \bar{c}^a c^b \big). \tag{A.2}$$

To evaluate the aforementioned two-point correlation function we expand the term $\mathrm{e}^{-S_{\mathrm{int}}}$ till the third order, so that

$$\langle \bar{c}^a(x) c^b(y) \rangle = \langle 0 | \bar{c}^a(x) c^b(y) \left(1 - S_{\mathrm{int}} + \frac{1}{2} S_{\mathrm{int}}^2 - \frac{1}{6} S_{\mathrm{int}}^3 + \cdots \right) | 0 \rangle$$

$$= \langle 0 | \bar{c}^a(x) c^b(y) | 0 \rangle - \langle 0 | \bar{c}^a(x) c^b(y) \, S_{\mathrm{int}} | 0 \rangle + \frac{1}{2} \langle 0 | \bar{c}^a(x) c^b(y) \, S_{\mathrm{int}}^2 | 0 \rangle$$

$$- \frac{1}{6} \langle 0 | \bar{c}^a(x) c^b(y) \, S_{\mathrm{int}}^3 | 0 \rangle + \cdots, \tag{A.3}$$

where

$$\langle 0 | \bar{c}^a(x) c^b(y) | 0 \rangle = \delta^{ab} G_0(x - y), \tag{A.4}$$

with

$$G_0(x - y) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2} \, \mathrm{e}^{\mathrm{i} q(x - y)}. \tag{A.5}$$

We are interested in terms of order 3 in the gauge field A or, equivalently, of order g^3. These terms appear in the expression for S_{int}^2 and S_{int}^3. Therefore, for the third-order correlation function $G_3^{ab}(x, y; A)$, one writes

$$G_3^{ab}(x, y; A) = \frac{1}{2} \Big\langle \bar{c}^a(x) c^b(y) \int \mathrm{d}^4 x_1 \, \mathrm{d}^4 x_2 \big(4g^3 f^{a_1 b_1 i_1} f^{a_2 c_2 i_2} f^{b_2 c_2 j_2} A_\mu^{i_1}(x_1) A_\nu^{i_2}(x_2) A_\nu^{j_2}(x_2)$$

$$+ 4g^3 f^{a_1 b_1 i_1} f^{a_2 d_2 c_2} f^{b_2 d_2 i_2} A_\mu^{i_1}(x_1) A_\nu^{c_2}(x_2) A_\nu^{i_2}(x_2)$$

$$- 4g^3 f^{a_1 b_1 i_1} f^{c_2 a_2 i_2} f^{d_2 b_2 i_2} A_\mu^{i_1}(x_1) A_\nu^{c_2}(x_2) A_\nu^{d_2}(x_2)$$

$$+ 2g^3 f^{a_1 b_1 c_1} f^{a_2 c_2 i_2} f^{b_2 c_2 j_2} A_\mu^{c_1}(x_1) A_\nu^{i_2}(x_2) A_\nu^{j_2}(x_2)$$

$$+ 2g^3 f^{a_1 b_1 c_1} f^{a_2 d_2 c_2} f^{b_2 d_2 i_2} A_\mu^{c_1}(x_1) A_\nu^{c_2}(x_2) A_\nu^{i_2}(x_2)$$

$$- 2g^3 f^{a_1 b_1 c_1} f^{c_2 a_2 i_2} f^{d_2 b_2 i_2} A_\mu^{c_1}(x_1) A_\nu^{c_2}(x_2) A_\nu^{d_2}(x_2) \big)$$

$$\times \bar{c}^{a_1}(x_1) \partial_\mu^{x_1} c^{b_1}(x_1) \bar{c}^{a_2}(x_2) c^{b_2}(x_2) \Big\rangle$$

$$
+ \frac{1}{6} \Big\langle \bar{c}^a(x) c^b(y) \int d^4x_1 \, d^4x_2 \, d^4x_3
$$

$$
\times \big(8g^3 f^{a_1 b_1 i_1} f^{a_2 b_2 i_2} f^{a_3 b_3 i_3} A_\mu^{i_1}(x_1) A_\nu^{i_2}(x_2) A_\sigma^{i_3}(x_3)
$$

$$
+ 12 g^3 f^{a_1 b_1 i_1} f^{a_2 b_2 i_2} f^{a_3 b_3 c_3} A_\mu^{i_1}(x_1) A_\nu^{i_2}(x_2) A_\sigma^{c_3}(x_3)
$$

$$
+ 6 g^3 f^{a_1 b_1 i_1} f^{a_2 b_2 c_2} f^{a_3 b_3 c_3} A_\mu^{i_1}(x_1) A_\nu^{c_2}(x_2) A_\sigma^{c_3}(x_3)
$$

$$
+ g^3 f^{a_1 b_1 c_1} f^{a_2 b_2 c_2} f^{a_3 b_3 c_3} A_\mu^{c_1}(x_1) A_\nu^{c_2}(x_2) A_\sigma^{c_3}(x_3) \big)
$$

$$
\times \bar{c}^{a_1}(x_1) \, \partial_\mu^{x_1} c^{b_1}(x_1) \, \bar{c}^{a_2}(x_2) \, \partial_\mu^{x_2} c^{b_2}(x_2) \, \bar{c}^{a_3}(x_3) \, \partial_\mu^{x_3} c^{b_3}(x_3) \Big\rangle . \tag{A.6}
$$

Performing the Wick contractions, we obtain

$$
G_3^{ab}(x, y; A) = \frac{1}{2} \int d^4x_1 \, d^4x_2 \big(4g^3 f^{a_1 b_1 i_1} f^{a_2 c_2 i_2} f^{b_2 c_2 j_2} A_\mu^{i_1}(x_1) A_\nu^{i_2}(x_2) A_\nu^{j_2}(x_2)
$$

$$
+ 4g^3 f^{a_1 b_1 i_1} f^{a_2 d_2 c_2} f^{b_2 d_2 i_2} A_\mu^{i_1}(x_1) A_\nu^{c_2}(x_2) A_\nu^{i_2}(x_2)
$$

$$
- 4g^3 f^{a_1 b_1 i_1} f^{c_2 a_2 i_2} f^{d_2 b_2 i_2} A_\mu^{i_1}(x_1) A_\nu^{c_2}(x_2) A_\nu^{d_2}(x_2)
$$

$$
+ 2g^3 f^{a_1 b_1 c_1} f^{a_2 c_2 i_2} f^{b_2 c_2 j_2} A_\mu^{c_1}(x_1) A_\nu^{i_2}(x_2) A_\nu^{j_2}(x_2)
$$

$$
+ 2g^3 f^{a_1 b_1 c_1} f^{a_2 d_2 c_2} f^{b_2 d_2 i_2} A_\mu^{c_1}(x_1) A_\nu^{c_2}(x_2) A_\nu^{i_2}(x_2)
$$

$$
- 2g^3 f^{a_1 b_1 c_1} f^{c_2 a_2 i_2} f^{d_2 b_2 i_2} A_\mu^{c_1}(x_1) A_\nu^{c_2}(x_2) A_\nu^{d_2}(x_2) \big)
$$

$$
\times \Big[\big(\partial_\mu^{x_1} \langle \bar{c}^a(x) c^{b_1}(x_1) \rangle \big) \langle \bar{c}^{a_2}(x_2) c^b(y) \rangle \langle \bar{c}^{a_1}(x_1) c^{b_2}(x_2) \rangle
$$

$$
+ \langle \bar{c}^a(x) c^{b_2}(x_2) \rangle \langle \bar{c}^{a_1}(x_1) c^b(y) \rangle \, \partial_\mu^{x_1} \langle \bar{c}^{a_2}(x_2) c^{b_1}(x_1) \rangle \Big]
$$

$$
- \frac{1}{6} \int d^4x_1 \, d^4x_2 \, d^4x_3 \big(8g^3 f^{a_1 b_1 i_1} f^{a_2 b_2 i_2} f^{a_3 b_3 i_3} A_\mu^{i_1}(x_1) A_\nu^{i_2}(x_2) A_\sigma^{i_3}(x_3)
$$

$$
+ 12 g^3 f^{a_1 b_1 i_1} f^{a_2 b_2 i_2} f^{a_3 b_3 c_3} A_\mu^{i_1}(x_1) A_\nu^{i_2}(x_2) A_\sigma^{c_3}(x_3)
$$

$$
+ 6 g^3 f^{a_1 b_1 i_1} f^{a_2 b_2 c_2} f^{a_3 b_3 c_3} A_\mu^{i_1}(x_1) A_\nu^{c_2}(x_2) A_\sigma^{c_3}(x_3)
$$

$$
+ g^3 f^{a_1 b_1 c_1} f^{a_2 b_2 c_2} f^{a_3 b_3 c_3} A_\mu^{c_1}(x_1) A_\nu^{c_2}(x_2) A_\sigma^{c_3}(x_3) \big)
$$

$$
\times \Big[\big(\partial_\mu^{x_1} \langle \bar{c}^a(x) c^{b_1}(x_1) \rangle \big) \langle \bar{c}^{a_2}(x_2) c^b(y) \rangle \big(\partial_\sigma^{x_3} \langle \bar{c}^{a_1}(x_1) c^{b_3}(x_3) \rangle \big) \partial_\nu^{x_2} \langle \bar{c}^{a_3}(x_3) c^{b_2}(x_2) \rangle
$$

$$
+ \big(\partial_\mu^{x_1} \langle \bar{c}^a(x) c^{b_1}(x_1) \rangle \big) \langle \bar{c}^{a_3}(x_3) c^b(y) \rangle \big(\partial_\nu^{x_2} \langle \bar{c}^{a_1}(x_1) c^{b_2}(x_2) \rangle \big) \partial_\sigma^{x_3} \langle \bar{c}^{a_2}(x_2) c^{b_3}(x_3) \rangle
$$

$$
+ \big(\partial_\nu^{x_2} \langle \bar{c}^a(x) c^{b_2}(x_2) \rangle \big) \langle \bar{c}^{a_1}(x_1) c^b(y) \rangle \big(\partial_\mu^{x_1} \langle \bar{c}^{a_3}(x_3) c^{b_1}(x_1) \rangle \big) \partial_\sigma^{x_3} \langle \bar{c}^{a_2}(x_2) c^{b_3}(x_3) \rangle
$$

$$
+ \big(\partial_\nu^{x_2} \langle \bar{c}^a(x) c^{b_2}(x_2) \rangle \big) \langle \bar{c}^{a_3}(x_3) c^b(y) \rangle \big(\partial_\sigma^{x_3} \langle \bar{c}^{a_1}(x_1) c^{b_3}(x_3) \rangle \big) \partial_\mu^{x_1} \langle \bar{c}^{a_2}(x_2) c^{b_1}(x_1) \rangle
$$

$$
+ \big(\partial_\sigma^{x_3} \langle \bar{c}^a(x) c^{b_3}(x_3) \rangle \big) \langle \bar{c}^{a_1}(x_1) c^b(y) \rangle \big(\partial_\mu^{x_1} \langle \bar{c}^{a_2}(x_2) c^{b_1}(x_1) \rangle \big) \partial_\nu^{x_2} \langle \bar{c}^{a_3}(x_3) c^{b_2}(x_2) \rangle
$$

$$
+ \big(\partial_\sigma^{x_3} \langle \bar{c}^a(x) c^{b_3}(x_3) \rangle \big) \langle \bar{c}^{a_2}(x_2) c^b(y) \rangle \big(\partial_\nu^{x_2} \langle \bar{c}^{a_1}(x_1) c^{b_2}(x_2) \rangle \big) \partial_\mu^{x_1} \langle \bar{c}^{a_3}(x_3) c^{b_1}(x_1) \rangle \Big] . \tag{A.7}
$$

Renaming now some dummy indices, it follows that

$$
G_3^{ab}(x, y; A) = \frac{1}{2} \int d^4x_1 \, d^4x_2 \big(4g^3 f^{a_1 b_1 i_1} f^{a_2 c_2 i_2} f^{b_2 c_2 j_2} A_\mu^{i_1}(x_1) A_\nu^{i_2}(x_2) A_\nu^{j_2}(x_2)
$$

$$
+ 4g^3 f^{a_1 b_1 i_1} f^{a_2 d_2 c_2} f^{b_2 d_2 i_2} A_\mu^{i_1}(x_1) A_\nu^{c_2}(x_2) A_\nu^{i_2}(x_2)
$$

$$
- 4g^3 f^{a_1 b_1 i_1} f^{c_2 a_2 i_2} f^{d_2 b_2 i_2} A_\mu^{i_1}(x_1) A_\nu^{c_2}(x_2) A_\nu^{d_2}(x_2)
$$

$$
+ 2g^3 f^{a_1 b_1 c_1} f^{a_2 c_2 i_2} f^{b_2 c_2 j_2} A_\mu^{c_1}(x_1) A_\nu^{i_2}(x_2) A_\nu^{j_2}(x_2)
$$

$$
+ 2g^3 f^{a_1 b_1 c_1} f^{a_2 d_2 c_2} f^{b_2 d_2 i_2} A_\mu^{c_1}(x_1) A_\nu^{c_2}(x_2) A_\nu^{i_2}(x_2)
$$

$$- 2g^3 f^{a_1 b_1 c_1} f^{c_2 a_2 i_2} f^{d_2 b_2 i_2} A_\mu^{c_1}(x_1) A_\nu^{c_2}(x_2) A_\nu^{d_2}(x_2) \big)$$

$$\times \big[\big(\partial_\mu^{x_1} \langle \bar{c}^a(x) c^{b_1}(x_1) \rangle \big) \langle \bar{c}^{a_2}(x_2) c^b(y) \rangle \langle \bar{c}^{a_1}(x_1) c^{b_2}(x_2) \rangle$$

$$+ \langle \bar{c}^a(x) c^{b_2}(x_2) \rangle \langle \bar{c}^{a_1}(x_1) c^b(y) \rangle \partial_\mu^{x_1} \langle \bar{c}^{a_2}(x_2) c^{b_1}(x_1) \rangle \big]$$

$$- \frac{1}{6} \int \mathrm{d}^4 x_1 \, \mathrm{d}^4 x_2 \, \mathrm{d}^4 x_3 \big(48 g^3 f^{a_1 b_1 i_1} f^{a_2 b_2 i_2} f^{a_3 b_3 i_3} A_\mu^{i_1}(x_1) A_\nu^{i_2}(x_2) A_\sigma^{i_3}(x_3)$$

$$+ 24 g^3 f^{a_1 b_1 i_1} f^{a_2 b_2 i_2} f^{a_3 b_3 c_3} A_\mu^{i_1}(x_1) A_\nu^{i_2}(x_2) A_\sigma^{c_3}(x_3)$$

$$+ 24 g^3 f^{a_1 b_1 i_1} f^{a_2 b_2 c_2} f^{a_3 b_3 i_3} A_\mu^{i_1}(x_1) A_\nu^{c_2}(x_2) A_\sigma^{i_3}(x_3)$$

$$+ 24 g^3 f^{a_1 b_1 c_1} f^{a_2 b_2 i_2} f^{a_3 b_3 i_3} A_\mu^{c_1}(x_1) A_\nu^{i_2}(x_2) A_\sigma^{i_3}(x_3)$$

$$+ 12 g^3 f^{a_1 b_1 i_1} f^{a_2 b_2 c_2} f^{a_3 b_3 c_3} A_\mu^{i_1}(x_1) A_\nu^{c_2}(x_2) A_\sigma^{c_3}(x_3)$$

$$+ 12 g^3 f^{a_1 b_1 c_1} f^{a_2 b_2 i_2} f^{a_3 b_3 c_3} A_\mu^{c_1}(x_1) A_\nu^{i_2}(x_2) A_\sigma^{c_3}(x_3)$$

$$+ 12 g^3 f^{a_1 b_1 c_1} f^{a_2 b_2 c_2} f^{a_3 b_3 i_3} A_\mu^{c_1}(x_1) A_\nu^{c_2}(x_2) A_\sigma^{i_3}(x_3)$$

$$+ 6 g^3 f^{a_1 b_1 c_1} f^{a_2 b_2 c_2} f^{a_3 b_3 c_3} A_\mu^{c_1}(x_1) A_\nu^{c_2}(x_2) A_\sigma^{c_3}(x_3) \big)$$

$$\times \big[\big(\partial_\mu^{x_1} \langle \bar{c}^a(x) c^{b_1}(x_1) \rangle \big) \langle \bar{c}^{a_2}(x_2) c^b(y) \rangle$$

$$\times \big(\partial_\sigma^{x_3} \langle \bar{c}^{a_1}(x_1) c^{b_3}(x_3) \rangle \big) \partial_\nu^{x_2} \langle \bar{c}^{a_3}(x_3) c^{b_2}(x_2) \rangle \big]. \tag{A.8}$$

It turns out to be convenient to introduce the following quantity:

$$\mathcal{G}^{(3)}(x, y; A) := \frac{1}{N(N-1)} G_3^{aa}(x, y; A). \tag{A.9}$$

Using expression (A.4), we obtain

$$\mathcal{G}^{(3)}(x, y; A) = - \frac{1}{N(N-1)} \int \mathrm{d}^4 x_1 \, \mathrm{d}^4 x_2 \, \mathrm{d}^4 x_3 \big(8 g^3 f^{abi} f^{bcj} f^{cak} A_\mu^i(x_1) A_\nu^j(x_2) A_\sigma^k(x_3)$$

$$+ 4 g^3 f^{abi} f^{bdj} f^{dac} A_\mu^i(x_1) A_\nu^j(x_2) A_\sigma^c(x_3)$$

$$+ 4 g^3 f^{abi} f^{bdc} f^{daj} A_\mu^i(x_1) A_\nu^c(x_2) A_\sigma^j(x_3)$$

$$+ 4 g^3 f^{abc} f^{bdi} f^{daj} A_\mu^c(x_1) A_\nu^i(x_2) A_\sigma^j(x_3)$$

$$+ 2 g^3 f^{abi} f^{bdc} f^{dae} A_\mu^i(x_1) A_\nu^c(x_2) A_\sigma^e(x_3)$$

$$+ 2 g^3 f^{abc} f^{bdi} f^{dae} A_\mu^c(x_1) A_\nu^i(x_2) A_\sigma^e(x_3)$$

$$+ 2 g^3 f^{abc} f^{bde} f^{dai} A_\mu^c(x_1) A_\nu^e(x_2) A_\sigma^i(x_3)$$

$$+ g^3 f^{abc} f^{bde} f^{daf} A_\mu^c(x_1) A_\nu^e(x_2) A_\sigma^f(x_3) \big)$$

$$\times \big[\big(\partial_\mu^{x_1} G_0(x - x_1) \big) G_0(x_2 - y) \big(\partial_\sigma^{x_3} G_0(x_1 - x_3) \big) \partial_\nu^{x_2} G_0(x_3 - x_2) \big]$$

$$+ \frac{1}{N(N-1)} \int \mathrm{d}^4 x_1 \, \mathrm{d}^4 x_2 \big(2 g^3 f^{abi} f^{bcj} f^{ack} A_\mu^i(x_1) A_\nu^j(x_2) A_\nu^k(x_2)$$

$$+ 2 g^3 f^{abi} f^{bdc} f^{adj} A_\mu^i(x_1) A_\nu^c(x_2) A_\nu^j(x_2)$$

$$- 2 g^3 f^{abi} f^{cbj} f^{daj} A_\mu^i(x_1) A_\nu^c(x_2) A_\nu^d(x_2)$$

$$+ g^3 f^{abc} f^{bdi} f^{adj} A_\mu^c(x_1) A_\nu^i(x_2) A_\nu^j(x_2)$$

$$+ g^3 f^{abc} f^{bde} f^{adi} A_\mu^c(x_1) A_\nu^e(x_2) A_\nu^i(x_2)$$

$$- g^3 f^{abc} f^{dbi} f^{eai} A_\mu^c(x_1) A_\nu^d(x_2) A_\nu^e(x_2) \big)$$

$$\times \big[\big(\partial_\mu^{x_1} G_0(x - x_1) \big) G_0(x_2 - y) G_0(x_1 - x_2)$$

$$+ G_0(x - x_2) G_0(x_1 - y) \partial_\mu^{x_1} G_0(x_2 - x_1) \big]. \tag{A.10}$$

Let us now show that almost all the terms in the double integrals vanish, i.e.

$$
\begin{aligned}
f^{abi} f^{bcj} f^{ack} A_\mu^i(x_1) A_\nu^j(x_2) A_\nu^k(x_2) &= -f^{abi} f^{acj} f^{bck} A_\mu^i(x_1) A_\nu^j(x_2) A_\nu^k(x_2)\\
&= -f^{abi} f^{ack} f^{bcj} A_\mu^i(x_1) A_\nu^k(x_2) A_\nu^j(x_2)\\
&= -f^{abi} f^{ack} f^{bcj} A_\mu^i(x_1) A_\nu^j(x_2) A_\nu^k(x_2)\\
&= 0,
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
f^{abi} f^{bdc} f^{adj} A_\mu^i(x_1) A_\nu^c(x_2) A_\nu^j(x_2) &= -f^{abi} f^{adc} f^{bdj} A_\mu^i(x_1) A_\nu^c(x_2) A_\nu^j(x_2)\\
&= -f^{abi}(-f^{adb} f^{jdc} - f^{adj} f^{cdb}) A_\mu^i(x_1) A_\nu^c(x_2) A_\nu^j(x_2)\\
&= (\underbrace{f^{abi} f^{adb}}_{-f^{abi} f^{abd}=0} f^{jdc} + f^{abi} f^{adj} f^{cdb}) A_\mu^i(x_1) A_\nu^c(x_2) A_\nu^j(x_2)\\
&= f^{abi} f^{adj} f^{cdb} A_\mu^i(x_1) A_\nu^c(x_2) A_\nu^j(x_2)\\
&= -f^{abi} f^{adj} f^{bdc} A_\mu^i(x_1) A_\nu^c(x_2) A_\nu^j(x_2)\\
&= 0,
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
f^{abi} f^{cbj} f^{daj} A_\mu^i(x_1) A_\nu^c(x_2) A_\nu^d(x_2) &= -f^{abi} f^{caj} f^{dbj} A_\mu^i(x_1) A_\nu^c(x_2) A_\nu^d(x_2)\\
&= -f^{abi} f^{daj} f^{cbj} A_\mu^i(x_1) A_\nu^d(x_2) A_\nu^c(x_2)\\
&= -f^{abi} f^{daj} f^{cbj} A_\mu^i(x_1) A_\nu^c(x_2) A_\nu^d(x_2)\\
&= 0,
\end{aligned}
\tag{A.13}
$$

$$
\begin{aligned}
f^{abc} f^{bdi} f^{adj} A_\mu^c(x_1) A_\nu^i(x_2) A_\nu^j(x_2) &= -f^{abc} f^{adi} f^{bdj} A_\mu^c(x_1) A_\nu^i(x_2) A_\nu^j(x_2)\\
&= -f^{abc} f^{adj} f^{bdi} A_\mu^c(x_1) A_\nu^j(x_2) A_\nu^i(x_2)\\
&= -f^{abc} f^{adj} f^{bdi} A_\mu^i(x_1) A_\nu^i(x_2) A_\nu^j(x_2)\\
&= 0,
\end{aligned}
\tag{A.14}
$$

$$
\begin{aligned}
f^{abc} f^{dbi} f^{eai} A_\mu^c(x_1) A_\nu^d(x_2) A_\nu^e(x_2) &= -f^{abc} f^{dai} f^{ebi} A_\mu^c(x_1) A_\nu^d(x_2) A_\nu^e(x_2)\\
&= -f^{abc} f^{eai} f^{dbi} A_\mu^c(x_1) A_\nu^e(x_2) A_\nu^d(x_2)\\
&= -f^{abc} f^{eai} f^{dbi} A_\mu^c(x_1) A_\nu^d(x_2) A_\nu^e(x_2)\\
&= 0.
\end{aligned}
\tag{A.15}
$$

The only term that does not vanish is $f^{abc} f^{bde} f^{adi} A_\mu^c(x_1) A_\nu^e(x_2) A_\nu^i(x_2)$. Thus, we can finally write $\mathcal{G}^{(3)}$ as

$$
\begin{aligned}
\mathcal{G}^{(3)}(x, y; A) = -\frac{1}{N(N-1)} \int \mathrm{d}^4 x_1\, \mathrm{d}^4 x_2\, \mathrm{d}^4 x_3 \big(&8g^3 f^{abi} f^{bcj} f^{cak} A_\mu^i(x_1) A_\nu^j(x_2) A_\sigma^k(x_3)\\
&+ 4g^3 f^{abi} f^{bdj} f^{dac} A_\mu^i(x_1) A_\nu^j(x_2) A_\sigma^c(x_3)\\
&+ 4g^3 f^{abi} f^{bdc} f^{daj} A_\mu^i(x_1) A_\nu^c(x_2) A_\sigma^j(x_3)\\
&+ 4g^3 f^{abc} f^{bdi} f^{daj} A_\mu^c(x_1) A_\nu^i(x_2) A_\sigma^j(x_3)\\
&+ 2g^3 f^{abi} f^{bdc} f^{dae} A_\mu^i(x_1) A_\nu^c(x_2) A_\sigma^e(x_3)\\
&+ 2g^3 f^{abc} f^{bdi} f^{dae} A_\mu^c(x_1) A_\nu^i(x_2) A_\sigma^e(x_3)\\
&+ 2g^3 f^{abc} f^{bde} f^{dai} A_\mu^c(x_1) A_\nu^e(x_2) A_\sigma^i(x_3)
\end{aligned}
$$

$$+ g^3 f^{abc} f^{bde} f^{daf} A_\mu^c(x_1) A_\nu^e(x_2) A_\sigma^f(x_3))$$
$$\times \left[\left(\partial_\mu^{x_1} G_0(x-x_1) \right) G_0(x_2-y) \left(\partial_\sigma^{x_3} G_0(x_1-x_3) \right) \partial_\nu^{x_2} G_0(x_3-x_2) \right]$$
$$+ \frac{1}{N(N-1)} \int \mathrm{d}^4 x_1 \, \mathrm{d}^4 x_2 \, g^3 f^{abc} f^{bde} f^{adi} A_\mu^c(x_1) A_\nu^e(x_2) A_\nu^i(x_2)$$
$$\times \left[\left(\partial_\mu^{x_1} G_0(x-x_1) \right) G_0(x_2-y) G_0(x_1-x_2) \right.$$
$$\left. + G_0(x-x_2) G_0(x_1-y) \, \partial_\mu^{x_1} G_0(x_2-x_1) \right]. \tag{A.16}$$

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